Continuous-Time Continuous-Valued Random Processes

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We focus on continuous-time continuous-valued RPs with the prototypical RP being the Gaussian process.

We start by recalling that a linear transformation of a vector RV with multivariate Normal distribution is multivariate normal. For example, a normal random vector $\vec{\boldsymbol{X}}$ with 0 means linearly transformed by the matrix T (i.e. $\vec{\boldsymbol{Y}} = T\vec{\boldsymbol{X}}$) has pdf

$$f_{\vec{\boldsymbol{Y}}}(\vec{\boldsymbol{y}}) = \frac{1}{|\det T|}\frac{1}{(2\pi)^{n/2}\sqrt{\det\Sigma}}e^{-\frac{1}{2}(T^{-1}\vec{\boldsymbol{y}})^\top\Sigma^{-1}(T^{-1}\vec{\boldsymbol{y}})} = \frac{1}{|\det T|}\frac{1}{(2\pi)^{n/2}\sqrt{\det\Sigma}}e^{-\frac{1}{2}\vec{\boldsymbol{y}}^\top {T^{-1}}^\top\Sigma^{-1}T^{-1}\vec{\boldsymbol{y}}}$$
$$= \frac{1}{(2\pi)^{n/2}\sqrt{\det T^\top\Sigma T}}e^{-\frac{1}{2}\vec{\boldsymbol{y}}^\top {T^{-1}}^\top\Sigma^{-1}T^{-1}\vec{\boldsymbol{y}}}$$

where in the last equality we used the fact that $\sqrt{\det T^\top\Sigma T} = \sqrt{\det T\ \det\Sigma\ \det T} = |\det T|\sqrt{\det\Sigma}$. It follows then that $T\vec{\boldsymbol{X}}$ is distributed multivariate normal with zero mean and $T^\top\Sigma T$ covariance matrix. A similar (but messier) proof holds for a linearly transformed random vector $T\vec{\boldsymbol{X}}$ whose mean vector is not zero.

Definition 1. *A random process $\mathcal{X} = \{X_t : t \in A \subset \mathbb{R}\}$ is a Gaussian process if all of its finite dimensional marginals are multivariate normal random vectors:*

$$\forall k > 0, \forall t_1, \ldots, t_k \in A, \quad f_{X_{t_1},\ldots,X_{t_k}}(\vec{\boldsymbol{x}}) = \frac{1}{(2\pi)^{k/2}\sqrt{\det\Sigma}}e^{(\vec{\boldsymbol{x}}-\vec{\boldsymbol{\mu}})^\top\Sigma^{-1}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{\mu}})}.$$

The mean vector and covariance matrix above may change depending on the selection of $k, t_1, \ldots, t_k$. However, they must be consistent in the sense described previously in class.

The mean function $m_{\mathcal{X}}(t)$ and the auto-covariance function $C_{\mathcal{X}}(t,s)$ define the mean of all the marginals RVs $X_{t_1}, \ldots, X_{t_k}$ as well as the covariance matrix and as a result they define the pdf for all the finite dimensional marginals of the Gaussian process. As a result, for a Gaussian process, the mean and auto-covariance function define it uniquely by Kolmogorov's theorem.

An interesting special case of the Gaussian process is the Wiener process.

Definition 2. *The Wiener process $\mathcal{Z} = \{Z_t, t \geq 0\}$ is a Gaussian process with $Z_0 = 0$ identically, $m_{\mathcal{Z}}(t) = 0$ and $C_{\mathcal{Z}}(t,s) = \alpha\min(t,s)$ for some $\alpha > 0$.*

We motivate the Wiener process by deriving it as the limit of the discrete-time discrete-valued random walk described in the previous handout, with step size h. Let $Z_t = \sum_{i=1}^n hY_n$, where $Y_n = 2X_n - 1$ and $X_1, X_2, \ldots$ be iid Bernoulli with $\theta = 1/2$ and $h > 0$. More precisely, the time interval $[0,t]$ is divided to n segments of length δ and Z_t counts the variables Y_n multiplied by a step size $h > 0$. Recall (from previous lecture) that $\mathsf{E}(Y_n) = \mathsf{E}(2X_n - 1) = 2\theta - 1 = 0$ and $\mathsf{Var}(Y_n) = \mathsf{Var}(2X_n - 1) = 4\mathsf{Var}(X_n) = 4\theta(1-\theta) = 1$. As a result we have $\mathsf{E}(hY_n) = 0$ and $\mathsf{Var}(hY_n) = h^2$.

We consider the above process in the limit $\delta \to 0, h \to 0, h = \sqrt{\alpha\delta}$ (for some $\alpha > 0$). In other words, the length of the segment goes to zero, the step size h goes to 0 while maintaining $h = \sqrt{\alpha\delta}$.

The Wiener process Z_t may be written as

$$Z_t = \lim_{\delta\to 0, h\to 0, h=\sqrt{\alpha\delta}} \sum_{i=1}^n hY_n = \lim_{n\to\infty}\sqrt{\alpha}\sqrt{\delta}\sum_{i=1}^n Y_n = \lim_{n\to\infty}\sqrt{\alpha}\sqrt{t/n}\sum_{i=1}^n Y_n$$
$$= \lim_{n\to\infty}\sqrt{\alpha t}\frac{\sum_{i=1}^n Y_n}{\sqrt{n}} = \sqrt{\alpha t}\lim_{n\to\infty}\frac{\sum_{i=1}^n Y_n}{\sqrt{n}}$$

where the third inequality comes from the fact that since the segment $[0,t]$ is divided to n segments of length δ, $n = \lfloor t/\delta \rfloor$ and $\delta = t/n$ (this is not an entirely rigorous argument - since we ignore the floor function - but we leave it at that.) Finally, by the central limit theorem $\lim_{n\to\infty} \frac{\sum_{i=1}^n Y_n}{\sqrt{n}}$ approaches a normal RV with 0 mean and 1 variance. Since Z_t is $\sqrt{\alpha t}$ times that normal RV, we have that Z_t is a normal RV with mean 0 and variance αt.

So far we showed that Z_t is a uni-variate normal RV. To show that it fulfills the above definition for the Wiener process we need to show (1) that it is a Gaussian process and (2) derive its auto-covariance function.

Consider the above derivation of Z_t as a random walk. Since Z_t is a counting process, its increments are independent and have the same distribution: i.e. $f_{Z_5 - Z_3} = f_{Z_2}$. As a result, the pdf of a finite dimensional marginal $f_{Z_{t_1},\ldots,Z_{t_k}}$ has multivariate Gaussian distribution which is a product of Gaussians

$$f_{Z_{t_1},\ldots,Z_{t_k}}(\vec{z}) = f_{Z_{t_1}}(z_1) f_{Z_{t_2}-Z_{t_1}}(z_2 - z_1) \cdots f_{Z_{t_k}-Z_{t_{k-1}}}(z_k - z_{k-1}).$$

The mean function $m_{\mathcal{Z}}(t) = 0$ as it is the sum of zero mean RVs. The auto-covariance is

$$C_{\mathcal{Z}}(t,s) = \mathsf{E}\left(\left(\lim \sum_{i=1}^{\lfloor t/\delta \rfloor} hY_i\right)\left(\lim \sum_{i=1}^{\lfloor s/\delta \rfloor} hY_i\right)\right) = \lim h^2 \mathsf{E}\left(\sum_{i=1}^{\lfloor t/\delta \rfloor}\sum_{j=1}^{\lfloor s/\delta \rfloor} Y_i Y_j\right) = \lim h^2 \sum_{i=1}^{\lfloor t/\delta \rfloor}\sum_{j=1}^{\lfloor s/\delta \rfloor} \mathsf{E}(Y_i Y_j).$$

In the above sum, $\mathsf{E}(Y_i Y_j) = 0$ for $i \neq j$ (since Y_i, Y_j are independent with mean 0) and the auto-covariance equals

$$C_{\mathcal{Z}}(t,s) = \lim h^2 \sum_{i=1}^{\lfloor \min(s,t)/\delta \rfloor} \mathsf{E}(Y_i^2) = \lim h^2 \min(s,t) \mathsf{Var}(Y)/\delta = \lim \alpha\delta \min(s,t)/\delta = \alpha \min(s,t)$$

where the second equality is justified by the limit process and the third equality follows from the fact that we take the limit at $h = \sqrt{\alpha\delta}$.