Continuous-Time Discrete-Valued Random Processes

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We focus on continuous-time discrete-valued RPs indexed by the non-negative reals $\mathcal{X} = \{X_t : t \geq 0\}$. The prototypical RP here is the Poisson process. We start by recalling the Poisson RV.

The Poisson RV X is a discrete RV, parameterized by $\alpha > 0$, with the following pmf: $p_X(k) = \frac{\alpha^k}{k!}e^{-\alpha}$ for k being a non-negative integer and 0 otherwise. It has mean and variance are equal to α. A Binomial RV with $n \to \infty, \theta \to 0$ approaches a Poisson RV with parameter $\alpha = n\theta$.

Definition 1. *A counting process $\mathcal{N} = \{N_t : t \geq 0\}$ is a Poisson process with rate λ if*

1. $N_0 = 0$
2. *$\mathcal{N}$ has independent increments.*
3. *$N_t - N_s$ is distributed Poisson with parameter $\lambda(t-s)$ for $0 \leq s < t < \infty$.*

The RVs N_t of the Poisson RP count the number of occurrences or arrivals of events or like phone calls arriving at switchboard or cars arriving at intersection until time t. The RV N_t has a Poisson distribution with parameter λt. The number of appearances between time t and s is Poisson with parameter $\lambda(t-s)$. The parameter λ describes the intensity of the occurrences or arrivals.

Example: Since the increments are independent and have Poisson distribution we have (assuming $t > s$)

$$\begin{aligned} P(N_s = i, N_t = j) &= P(N_s - N_0 = i - 0 \cap N_t - N_s = j - i) = P(N_s = i)P(N_t - N_s = j - i) \\ &= \frac{(\lambda s)^i}{i!}e^{-\lambda s}\frac{(\lambda(t-s))^{j-i}}{(j-i)!}e^{-\lambda(t-s)}. \end{aligned}$$

To motivate the Poisson RP we relate it to a discrete-time counting process of Bernoulli iid RVs (see previous handout). Let the time interval $[0,t]$ be divided to n equally sized parts, and consider $n \to \infty$. Assume that we have n iid Bernoulli RVs each describing whether an event occurred in one of the above segments. Since the size of each of the n intervals is extremely small (as $n \to \infty$) it is sufficient to consider Bernoulli RVs (1 arrival or 0) and ignore the possibility of two or more events in a single segment. The probability that two arrivals or occurrences will happen in such a small time interval is negligent. The distribution of the number of events by time t is Binomially distributed with parameters n, θ and expectation $n\theta$. As $n \to \infty, \theta \to 0$, the number of occurrences approaches a Poisson distribution by the above approximation result with parameter $\alpha = n\theta$. Since the occurrences of the Bernoulli RVs are independent of each other, the counting process has independent increments. If we count in a similar fashion the number of occurrences over several segments of length t we need to multiply the rate α by the length of the several segments αt.

The mean function of the Poisson process is $m_{\mathcal{N}}(t) = \lambda t$ (note that the mean increases with time, as it should for a counting process) and the auto-covariance is (assume below $t \geq s$)

$$\begin{aligned} C_{\mathcal{N}}(t,s) &= \mathsf{E}((N_s - \lambda s)(N_t - \lambda t)) = \mathsf{E}((N_s - \lambda s)((N_t - N_s - \lambda t + \lambda s) + (N_s - \lambda s))) \\ &= \mathsf{Var}(N_s) + \mathsf{E}((N_s - \lambda s)(N_t - N_s - \lambda t + \lambda s)) = \mathsf{Var}(N_s) + \mathsf{E}(N_s - \lambda s)\mathsf{E}(N_t - N_s - \lambda t + \lambda s) \\ &= \mathsf{Var}(N_s) = \lambda s = \lambda \min(s,t). \end{aligned}$$

Proposition 1. *The times between subsequent arrivals or events in the Poisson process are independent exponential RVs with parameter λ and mean $1/\lambda$.*

In particular, the time of the first arrival is exponentially distributed, and so is the time between the k and the $k+1$ arrival.

Another interesting fact is that given that one arrival has occurred in the interval $[0,t]$, the precise time x of that arrival is uniformly distributed in that interval: we have the probability that the arrival was before time x

$$\begin{aligned} P(N_x = 1 | N_t = 1) &= \frac{P(N_x = 1 \cap N_t = 1)}{P(N_t = 1)} = \frac{P(N_x = 1 \cap N_t - N_x = 0)}{P(N_t = 1)} \\ &= \frac{P(N_x = 1) P(N_t - N_x = 0)}{P(N_t = 1)} = \frac{(\lambda x) e^{-\lambda x}}{1} \frac{(\lambda (t-x))^0 e^{-\lambda (t-x)}}{1} \frac{1}{(\lambda t)^1 e^{-\lambda t}} = \frac{x}{t} \end{aligned}$$

which is the the cdf of the uniform distribution on the interval $[0,t]$. In that sense the arrival or occurrences of the Poisson process are truly at random.

We can write a realization of the Poisson process as a sum of shifted step functions

$$N_\omega(t) = \sum_{t=1}^{\infty} u(t - S_i)$$

where S_i are the locations of the arrivals or event occurrences. If we define a process $\mathcal{S} = \{S_t, t \geq 0\}$ whose realizations are impulse functions shifted by the arrivals described by a Poisson RP, we can express the Poisson process as filtering of the RP $\mathcal{S}$ through a linear system whose impulse response is a step function. Furthermore, additional processes may be defined by filtering the same impulse train process $\mathcal{S}$ through different linear systems. For example, we can define $\mathcal{R} = \{R_t : t \geq 0\}$ as a the filtered RP of $\mathcal{S}$ going through a linear system with impulse response h

$$R_t = \sum_{i=1}^{\infty} h(t - S_i)$$

The shape of the impulse response of the system will determine the details of the filtered process. If the impulse response describes the current that is initiated by an external event, and if the counting process for the external events is a Poisson RP, then the total current flowing is $\mathcal{S}$ filtered through the linear system.