# Discrete-Time Discrete-Valued Random Processes

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We focus on discrete-time discrete-valued RP indexed by the positive integers: $\mathcal{X} = \{X_n : n = 1, 2, \ldots\}$.

The simplest example of such a RP is where $X_n$ are iid. In this case, the mean function is a constant function $m_{\mathcal{X}}(n) = \mathsf{E}(X_n) = \mathsf{E}(X_1) = \mu$ and the auto-covariance is

$$C_{\mathcal{X}}(n_1, n_2) = \mathsf{E}((X_{n_1} - \mu)(X_{n_2} - \mu)) = \begin{cases} \mathsf{E}(X_{n_1} - \mathsf{E}(X_{n_1}))\mathsf{E}(X_{n_2} - \mathsf{E}(X_{n_2})) = 0 & n_1 \neq n_2 \\ \mathsf{Var}(X_{n_1}) = \mathsf{Var}(X_1) & n_1 = n_2 \end{cases}$$

and so $C_{\mathcal{X}}(n_1, n_2) = \delta_{n_1,n_2} \mathsf{Var}(X_1)$. The auto-correlation is

$$R_{\mathcal{X}}(n_1, n_2) = C_{\mathcal{X}}(n_1, n_2) + m_{\mathcal{X}}(n_1) m_{\mathcal{X}}(n_2) = \delta_{n_1,n_2} \mathsf{Var}(X_1) + (\mathsf{E}(X_1))^2.$$

If $X_n$ are iid Bernoulli with parameter $\theta$ they each have

$$\mathsf{E}(X_n) = \theta \cdot 1 + (1 - \theta) \cdot 0 = \theta$$

$$\mathsf{Var}(X_n) = \mathsf{E}(X_n^2) - (\mathsf{E}(X_n))^2 = \theta \cdot 1^2 + (1 - \theta) \cdot 0^2 - \theta^2 = \theta(1 - \theta).$$

and we have $m_{\mathcal{X}}(n_1) = \theta$, $C_{\mathcal{X}}(n_1, n_2) = \delta_{n_1,n_2} \theta(1 - \theta)$, $R_{\mathcal{X}}(n_1, n_2) = \delta_{n_1,n_2} \theta(1 - \theta) + \theta^2$.

Alternatively, consider the RP $\mathcal{Y} = \{Y_n : n = 1, 2, \ldots\}$ where $Y_n$ are iid $Y_n = 2X_n - 1$ (where $X_n$ are iid Bernoulli as above). The RV $Y_n$ take on values 1 with probability $\theta$ and -1 with probability $1 - \theta$. They have $\mathsf{E}(Y_n) = \mathsf{E}(2X_n - 1) = 2\theta - 1$ and $\mathsf{Var}(Y_n) = \mathsf{Var}(2X_n - 1) = 4\mathsf{Var}(X_n) = 4\theta(1 - \theta)$. The RP $\mathcal{Y}$ has $m_{\mathcal{Y}}(n_1) = 2\theta - 1$, $C_{\mathcal{Y}}(n_1, n_2) = \delta_{n_1,n_2} 4\theta(1 - \theta)$, $R_{\mathcal{Y}}(n_1, n_2) = \delta_{n_1,n_2} 4\theta(1 - \theta) + (2\theta - 1)^2$.

We now turn to an example of a more interesting process - one that does not consist of independent RVs. We define the RP $\mathcal{W} = \{W_n : n = 1, 2, \ldots\}$, $W_n = \sum_{i=1}^n X_n$ and $\mathcal{Z} = \{Z_n : n = 1, 2, \ldots\}$, $Z_n = \sum_{i=1}^n Y_n$. These processes are called counting processes since they count the RVs of another process. These counting processes have dependent RVs since for example $W_n = W_{n-1} + X_n$. However, the counting processes have independent increments, for example $W_5 - W_3 = X_4 + X_5$ and $W_9 - W_7 = X_8 + X_9$ are independent RVs as they are sums of iid RVs.

The RP $\mathcal{Z}$ may be imagined as a one-dimensional discrete random walk. At any time $1, 2, \ldots$ the random walker decides whether to walk forward or backwards one step. The walker's position after time $n$ is described by $W_n$. The mean function is $m_{\mathcal{W}}(n) = \mathsf{E}(\sum_{i=1}^n Y_i) = \sum_{i=1}^n \mathsf{E}(Y_i) = n(2\theta - 1)$ and the auto-covariance is

$$C_{\mathcal{W}}(n, k) = \mathsf{E}\left( \left( \sum_{i=1}^n (Y_i - \mathsf{E}(Y_1)) \right) \left( \sum_{j=1}^k (Y_j - \mathsf{E}(Y_1)) \right) \right) = \mathsf{E}\left( \sum_{i=1}^n \sum_{j=1}^k (Y_i - \mathsf{E}(Y_1))(Y_j - \mathsf{E}(Y_1)) \right)$$

$$= \sum_{i=1}^n \sum_{i=1}^k \mathsf{E}((Y_i - \mathsf{E}(Y_1))(Y_j - \mathsf{E}(Y_1))) = \sum_{i=1}^n \sum_{i=1}^k \delta_{n,k} 4\theta(1 - \theta) = \min(k, n) 4\theta(1 - \theta).$$

Next, we derive the pmf of $Z_n$. For $Z_n = r$ we need to have $k$ of the $(Y_1, \ldots, Y_n)$ to be 1 and $n - k$ of them to be -1, where we require $k - (n - k) = r$ or $2k - n = r$ which holds only if $k = (r + n)/2$. The probability of that is easily computed as a binomial probability

$$p_{Z_n}(r) = \binom{n}{(r + n)/2} \theta^{(r+n)/2} (1 - \theta)^{n-(r+n)/2}$$

if $(r + n)/2$ is an non-negative integer less than or equal to $n$ and $p_{Z_n}(r) = 0$ otherwise.