

Random Processes - Basic Definitions

Guy Lebanon

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We started some time ago with random variables $X : \Omega \rightarrow \mathbb{R}$ and moved to random vectors (X_1, \dots, X_n) which are a vector of random variables. In a similar way, random processes are an infinite collection of random variables (of infinite dimensional random vectors). Discrete-time random processes are a collection of RV indexed by a countable set, for example the natural numbers $\mathcal{X} = \{X_n : n = 1, 2, \dots\}$ or the integers $\mathcal{X} = \{X_n : n \in \mathbb{Z}\}$. Continuous-time processes are a collection of RV indexed by a non-countable set that is a subset of the real-line, for example $\mathcal{X} = \{X_t : t \in \mathbb{R}, t \geq 0\}$. We will use the index n for discrete-time processes and t for continuous-time processes. When we want to treat both cases we will use the index t . Regardless of whether the process \mathcal{X} are discrete-time or continuous-time the RVs X_n or X_t can be either discrete or continuous RVs.

If we look at the random processes as a function of the index n (for discrete-time) or t (for continuous-time) and hold $\omega \in \Omega$ fixed we get a discrete-time or a continuous-time signal (or function) X_ω , e.g. $X_\omega(t) = X_t(\omega)$, called the realization or sample path of the processes at $\omega \in \Omega$. In this view, random processes are probabilistic models on continuous or discrete signals. Just like vector-RV we can define events

$$\{\mathcal{X} \in \mathcal{A}\} = \{\omega \in \Omega : X_\omega \in \mathcal{A}\}$$

for \mathcal{A} being a set of (discrete-time or continuous-time) functions. The probability $P(\mathcal{X} \in \mathcal{A})$ is then defined as the probability of that event. The processes can be independent (meaning the collection of RV defining the process are independent) or not.

Example (independent discrete-time discrete-valued processes): let $\Omega = (0, 1)$ and P be a uniform distribution over it. Denoting b_1, b_2, \dots as the decimal expansion of ω i.e. $\omega = \sum_{i=1}^{\infty} b_i 10^{-i}$, $b_i \in \{0, 1, \dots, 9\}$ we define the processes \mathcal{X} by $X_n(\omega) = b_n$, $X_\omega = (b_1, b_2, \dots)$ and for example, $P(\mathcal{X} = (1, 1, 1, 1, \dots)) = \prod_{n=1}^{\infty} P(X_n = 1) = \prod_{n=1}^{\infty} \frac{1}{10} = 0$.

Example (dependent continuous-time continuous-valued processes): For the same sample space and P , consider the process defined by $X_t(\omega) = \omega \sin(2\pi t)$. The realization path is a sinusoid with random amplitude. This is a dependent process ($X_\omega(t_1)$ and $X_\omega(t_2)$ are dependent RVs) with continuous-time and continuous RVs.

Just as we define the marginal distribution of a random vector, we can define the finite-dimensional marginal distribution of a random process, for example,

$$P((X_1, \dots, X_k) \in A) = P(\{\omega \in \Omega : (X_1(\omega), \dots, X_k(\omega)) \in A\}).$$

The notion of pdf and pmf are difficult to translate to random processes due to its infinite dimensionality. Instead, we specify them for the finite-dimensional marginals of the processes.

Theorem 1 (Kolmogorov). *Let $I \subset \mathbb{R}$ be an index set (think of this as the range of the index n or t). A collection of cdfs $F_{X_{c_1}, \dots, X_{c_k}}(x_{c_1}, \dots, x_{c_k})$ for all choices of $c_1, \dots, c_k \in I$ and for all orders $k = 1, 2, \dots$ that is consistent in the sense that*

$$F_{X_{c_1}, \dots, X_{c_k}}(x_{c_1}, \dots, x_{c_k}) = F_{X_{c_1}, \dots, X_{c_k}, X_{c_{k+1}}}(x_{c_1}, \dots, x_{c_k}, \infty)$$

uniquely defines a random process for which the cdfs above are the marginal cdfs.

In a similar way, a collection of pdfs $f_{X_{c_1}, \dots, X_{c_k}}(x_{c_1}, \dots, x_{c_k})$ for all choices of c_1, \dots, c_k and for all orders k uniquely defines a continuous-valued process if the pdf are consistent in the sense that

$$\int_{-\infty}^{\infty} f_{X_{c_1}, \dots, X_{c_k}, X_{c_{k+1}}}(x_{c_1}, \dots, x_{c_k}, x_{c_{k+1}}) dx_{c_{k+1}} = f_{X_{c_1}, \dots, X_{c_k}}(x_{c_1}, \dots, x_{c_k}).$$

A collection of pmfs $p_{X_{c_1}, \dots, X_{c_k}}(x_{c_1}, \dots, x_{c_k})$ for all choices of c_1, \dots, c_k and for all orders k uniquely defines a discrete-valued process if the pmfs are consistent in the sense

$$\sum_{x_{c_{k+1}}} p_{X_{c_1}, \dots, X_{c_k}, X_{c_{k+1}}}(x_{c_1}, \dots, x_{c_k}, x_{c_{k+1}}) = p_{X_{c_1}, \dots, X_{c_k}}(x_{c_1}, \dots, x_{c_k}).$$

The standard relation between these functions (cdf, pmf, pdf) behaves as described earlier in the course for finite dimensional random vectors).

Example: The discrete-time process described above can be defined by the collection of k -order pmfs

$$P(X_{c_1} = x_{c_1}, \dots, X_{c_k} = x_{c_k}) = \frac{1}{10^k}$$

for all k and for all $c_1, \dots, c_k \in \{0, 1, \dots, 9\}$. Note that the pmfs are consistent with each other (in the sense described above).

Definition 1. A process is said to have independent increments if for any k and for all choice of indices $t_1 < \dots < t_k$ the random variables

$$X_{t_2} - X_{t_1}, \dots, X_{t_k} - X_{t_{k-1}}$$

are independent (this definition holds for both discrete-time and continuous-time).

Definition 2. A process is said to be a Markov process if for any k and for all choice of indices $t_1 < \dots < t_k$

$$f_{X_{t_k} | X_{t_1} = x_{t_1}, \dots, X_{t_{k-1}} = x_{t_{k-1}}}(x_{t_k}) = f_{X_{t_k} | X_{t_{k-1}} = x_{t_{k-1}}}(x_{t_k}) \text{ for continuous-time processes} \quad (1)$$

$$p_{X_{t_k} | X_{t_1} = x_{t_1}, \dots, X_{t_{k-1}} = x_{t_{k-1}}}(x_{t_k}) = p_{X_{t_k} | X_{t_{k-1}} = x_{t_{k-1}}}(x_{t_k}) \text{ for discrete-time processes} \quad (2)$$

A process with independent increments is necessarily Markov but not vice-versa.

Recall that for vector RV the mean and covariance are vectors and matrices. The analogue for processes become functions. Note however, the inappropriate (but conventional) definition of autocorrelation in light of our previous definition of correlation.

- The mean of a random process \mathcal{X} is the function $m_{\mathcal{X}}(t) = \mathbf{E}(X_t)$
- The autocorrelation of a random process \mathcal{X} is the function $R_{\mathcal{X}}(t_1, t_2) = \mathbf{E}(X_{t_1} X_{t_2})$
- The auto-covariance of a random process \mathcal{X} is the function $C_{\mathcal{X}}(t_1, t_2) = \mathbf{E}((X_{t_1} - m_{\mathcal{X}}(t_1))(X_{t_2} - m_{\mathcal{X}}(t_2)))$. By the results we have for covariance we also have $C_{\mathcal{X}}(t_1, t_2) = R_{\mathcal{X}}(t_1, t_2) - m_{\mathcal{X}}(t_1)m_{\mathcal{X}}(t_2)$.
- The variance at time t of the process is $C_{\mathcal{X}}(t, t)$

Example: For a RV Y define a process $X_t = Y \cos(2\pi t)$. The mean is $m_{\mathcal{X}}(t) = \mathbf{E}(Y \cos(2\pi t)) = \mathbf{E}(Y) \cos(2\pi t)$ and the auto-correlation is $R_{\mathcal{X}}(t_1, t_2) = \mathbf{E}(Y \cos(2\pi t_1) Y \cos(2\pi t_2)) = \mathbf{E}(Y^2) \cos(2\pi t_1) \cos(2\pi t_2)$ and the autocovariance is

$$C_{\mathcal{X}}(t_1, t_2) = R_{\mathcal{X}}(t_1, t_2) - m_{\mathcal{X}}(t_1)m_{\mathcal{X}}(t_2) = (\mathbf{E}(Y^2) - (\mathbf{E}(Y))^2) \cos(2\pi t_1) \cos(2\pi t_2)$$

If we have two processes \mathcal{X}, \mathcal{Y} we have similar definitions for their interaction:

- The two processes are said to be independent if all the finite dimensional marginals of the two processes $(X_{t_1}, \dots, X_{t_k}), (Y_{t'_1}, \dots, Y_{t'_k})$ are two vector RVs (independent from each other).
- The cross correlation is $R_{\mathcal{X}, \mathcal{Y}}(t_1, t_2) = \mathbf{E}(X_{t_1} Y_{t_2})$. If the cross correlation is identically zero the two processes are said to be orthogonal.
- The cross-covariance of the two processes is $C_{\mathcal{X}, \mathcal{Y}}(t_1, t_2) = \mathbf{E}((X_{t_1} - m_{\mathcal{X}}(t_1))(Y_{t_2} - m_{\mathcal{Y}}(t_2)))$.