Affine Connections and Covariant Derivatives

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We summarize some facts concerning affine connections in differential geometry and covariant derivatives, based on the description in Chapter 1 of [1]. If the manifold M is embedded in $\mathbb{R}^m$, there is a natural correspondence between the tangent spaces T_pM, T_qM of nearby points p, q. In general, however, there is no such natural correspondence. The notion of connection formalizes this correspondence between tangent spaces. Many different connections or correspondences can be applied to a manifold M, each resulting in a different geometric structure. This geometric structure is separate from the Riemannian notion of distance and is associated with flatness, curvature and parallelness. A covariant derivative defines a differentiation of vector fields on a manifold. It is equivalent to a connection, and leads to the same geometric structure. We start below by defining a connection, and then proceed to define covariant derivatives and examine the relationship of these two concepts.

A connection Π is a mapping $\Pi_{p,q} : T_pM \to T_qM$, along a specific path connecting p, q. If p is close to q, we can assume that the map Π is in a linearized form (via local Taylor expansion)

$$\Pi_{p,q}(\partial_j) = \partial'_j - \sum_i (\xi_i(q) - \xi_i(p)) \sum_k (\Gamma^k_{ij})_p \ \partial'_k \tag{1}$$

$$\Pi_{p,q}\left(\sum_j Y^j \partial_j\right) = \sum_j Y^j \partial'_j - \sum_j Y^j \sum_i (\xi_i(q) - \xi_i(p)) \sum_k (\Gamma^k_{ij})_p \, \partial'_k \tag{2}$$

$$\Pi_{\gamma(t),\gamma(t+dt)}\left(\sum_j Y^j(t) \partial_j\right) = \sum_j Y^j(t) \partial'_j - \sum_j Y^j(t) \sum_i dt \dot{\gamma}^i(t) \sum_k (\Gamma^k_{ij})_{\gamma(t)} \, \partial'_k \tag{3}$$

where $\partial_j \in T_pM, \partial'_j \in T_qM$ and ξ is a local chart. The functions $\Gamma^k_{ij} \in C^\infty(M)$ determine the local coefficients that define the connection.

Above, Π defines a linear isomorphism between tangent spaces of neighboring points. We can also define a linear isomorphism between tangent spaces of two arbitrary points by integrating along a curve connecting the two points. This defines a vector field X along the curve γ that satisfies $X(t + dt) = \Pi_{\gamma(t),\gamma(t+dt)}(X(t))$. In this case we say that the vector field is parallel along the curve γ with respect to the selected connection. In other words, a vector field is parallel across a curve, if it changes precisely according to its parallel transport (defined by the connection). A vector field is parallel in a region if it is parallel with respect to any possible curve in the region.

Definition 1. *The covariant derivative ∇ of a vector field Y in the direction of a vector field X is a vector field $\nabla_X Y$ that is uniquely defined by the following properties*

1. $\nabla_{f_1X_1+f_2X_2} Y = f_1\nabla_{X_1} Y + f_2\nabla_{X_2} Y$
2. $\nabla_X(Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$
3. $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$.

Note that a linear combination of covariant derivatives is also a covariant derivative.

The covariant derivative is related to the connection as the deviation of the vector field along a curve from its parallel transport

$$\nabla_{\dot\gamma(t)} Y = \lim_{dt\to 0} \frac{\sum_j Y^j(t+dt)\partial'_j - \Pi_{\gamma(t),\gamma(t+dt)}\left(\sum_j Y^j(t)\partial_j\right)}{dt} \tag{4}$$

$$= \lim_{dt\to 0} \frac{\sum_j Y^j(t+dt)\partial'_j - \sum_j Y^j(t)\partial'_j + \sum_j Y^j(t)\sum_i dt\dot\gamma^i(t)\sum_k \Gamma^k_{ij}\,\partial'_k}{dt} \tag{5}$$

$$= \lim_{dt\to 0}\sum_j \frac{Y^j(t+dt)-Y^j(t)}{dt}\partial'_j + \sum_j Y^j(t)\sum_i \dot\gamma^i(t)\sum_k \Gamma^k_{ij}\,\partial'_k \tag{6}$$

$$= \sum_k (\dot\gamma(t)Y^k)\ \partial'_k + \sum_j Y^j(t)\sum_i \dot\gamma^i(t)\sum_k \Gamma^k_{ij}\,\partial'_k. \tag{7}$$

Replacing $\dot\gamma(t)$ and Y with vector fields we obtain the quantitative description of ∇ in terms of the connection coefficients:

$$\nabla_{\partial_i}\partial_j = \sum_k \Gamma^k_{ij}\partial_k$$

$$\nabla_{\partial_i} Y = \sum_j \nabla_{\partial_i} Y^j\partial_j = \sum_j Y^j\nabla_{\partial_i}\partial_j + (\partial_i Y^j)\partial_j = \sum_j Y^j\left(\sum_k \Gamma^k_{ij}\partial_k + (\partial_i Y^j)\partial_j\right) = \sum_k\left(\sum_j Y^j\Gamma^k_{ij} + \partial_i Y^k\right)\partial_k$$

$$\nabla_X Y = \sum_i X^i\sum_k\left(\sum_j Y^j\Gamma^k_{ij} + \partial_i Y^k\right)\partial_k.$$

In light of the above, we have

Definition 2. *A vector field Y in (M,∇) along a curve γ is said to be parallel if $\nabla_{\dot\gamma}Y = 0$, $\forall t$. If $\nabla_X Y = 0$, $\forall X$, then Y is said to be parallel. This is equivalent to $\sum_j Y^j\Gamma^k_{ij}\partial_k + \partial_i Y^k = 0 \quad \forall i\forall k$.*

Definition 3. *A curve γ on (M,∇) is a geodesic if $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$ for all t.*

Definition 4. *Let (M,∇) be a manifold with a certain coordinate system ξ. If the vector fields $\{\partial_i = \partial/\partial\xi^i\}$ are parallel, we say that ξ is an affine coordinate system (with respect to ∇) (this is equivalent to the requirement that $\nabla_{\partial_i}\partial_j = 0, \forall i,j$ or $\Gamma^k_{ij} = 0, \forall i,j,k$). If an affine coordinate system exists for (M,∇) we say that ∇ (or M with respect to ∇) is flat*

Definition 5. *For a manifold (M,g), the connection defined by $2\Gamma^k_{ij} = \partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}$ is called the metric connection. Under this connection, parallel transport of tangent vectors leaves their inner product unchanged. Since the inner product is invariant, the angle does not change either and the parallel transport is indeed "parallel". Furthermore the geodesic curves with respect to this connection are also shortest paths.*

Definition 6. *A submanifold N of (M,∇) is autoparallel if $\nabla_X Y \in TN$ for all $X, Y \in TN$. An equivalent condition is that parallel transport of tangent vectors for N w.r.t ∇ remain tangent vectors for N.*

References

[1] Shun-Ichi Amari and Hiroshi Nagaoka. *Methods of Information Geometry.* American Mathematical Society, 2000.