

LTI Filtering of WSS Processes

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We start with a brief reminder of LTI filtering. The Fourier transform of a function $g : \mathbb{R} \rightarrow \mathbb{R}$ is a function $\mathcal{F}(g) : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\mathcal{F}(g)(f) = G(f) = \int_{\mathbb{R}} g(t)e^{-2\pi ift} dt$$

and the inverse Fourier transform of $G(f)$ is

$$\mathcal{F}^{-1}(G)(t) = g(t) = \int_{\mathbb{R}} G(f)e^{2\pi ift} df.$$

There are other ways of defining the Fourier transform that differ slightly, including using the angular frequency ω instead of f . In contrast to the definition that uses ω , the form defined above has the advantage of symmetry between \mathcal{F} , \mathcal{F}^{-1} . In a similar way, the Fourier transform and its inverse for discrete-time signals is defined using sums instead of integrals. Note that the Fourier transform may in general be complex valued. An extremely useful feature of the Fourier representation is that it converts convolution into multiplication and vice versa

$$\mathcal{F}(g_1 \star g_2)(f) = \mathcal{F}(g_1)(f) \cdot \mathcal{F}(g_2)(f)$$

An LTI system is a linear time-invariant system i.e. $T[\alpha x_1(t) + \beta x_2(t)] = \alpha T[x_1(t)] + \beta T[x_2(t)]$ and $T[x(t - \tau)] = y(t - \tau)$ where $y(t) = T[x(t)]$. The impulse response $h(t)$ of an LTI system is its response to an impulse function $\delta(t) = T[\delta(t)]$. The Fourier transform of the impulse function is the transfer function $H(f)$. The output of an LTI system is a convolution of the input and the impulse response

$$T[x(t)] = \int_{-\infty}^{\infty} h(s)x(t - s)dx$$

(in the discrete-time case the convolution is a sum over all integers) and its Fourier transform is a multiplication of the Fourier representation of the signal with the transfer function $\mathcal{F}(T[x])(f) = H(f)\mathcal{F}(x)(f)$.

Definition 1. *The power spectral density $S_{\mathcal{X}}(f)$ of a WSS process \mathcal{X} is the Fourier transform of its autocorrelation function $R_{\mathcal{X}}(t)$.*

The power spectral density $S_{\mathcal{X}}$ is an even real-valued function as it is an integral of a multiplication of two even real-valued functions

$$\begin{aligned} S_{\mathcal{X}}(f) &= \int_{\mathbb{R}} R_{\mathcal{X}}(t)(\cos(2\pi ft) - i \sin(2\pi ft))dt = \int_{\mathbb{R}} R_{\mathcal{X}}(t) \cos(2\pi ft)dt - j \int_{\mathbb{R}} R_{\mathcal{X}}(t) \sin(2\pi ft)dt \\ &= \int_{\mathbb{R}} R_{\mathcal{X}}(t) \cos(2\pi ft)dt - 0 \end{aligned}$$

where we used the fact that an even function times an odd function is odd. A similar derivation holds for discrete time RPs.

As an example, consider the RP possessing a power spectral density $S_{\mathcal{X}}(f)$ that is uniform in $f \in [-a, a]$ and 0 otherwise. Such RP is called band limited white-noise. The autocorrelation of the RP may be recovered by the inverse Fourier transform and turns out to be the sinc function. If $a \rightarrow \infty$ the RP is just called white

noise. In this case the autocorrelation function approaches an impulse function. Recall that a Gaussian process is uniquely defined by its mean and auto-correlation function (or alternatively the mean and the power spectral density function). A popular modeling of noises is therefore zero-mean white-noise Gaussian RP.

Example: Let \mathcal{X} be an RP and \mathcal{Y} be white-noise zero-mean RP independent of \mathcal{X} . The corrupted process defined by $\mathcal{Z} = \mathcal{X} + \mathcal{Y}$ has

$$\begin{aligned} m_{\mathcal{Z}}(t) &= \mathbf{E}(X_t + Y_t) = \mathbf{E}(X_t) + \mathbf{E}(Y_t) = m_{\mathcal{X}}(t) \\ R_{\mathcal{Z}}(\tau) &= \mathbf{E}((X_t + Y_t)(X_{t+\tau} + Y_{t+\tau})) = R_{\mathcal{X}}(\tau) + \mathbf{E}(X_{t+\tau}Y_t) + \mathbf{E}(X_tY_{t+\tau}) + R_{\mathcal{Y}}(\tau) \\ &= R_{\mathcal{X}}(\tau) + \text{Cov}(X_{t+\tau}, Y_t) + \mathbf{E}(X_{t+\tau})\mathbf{E}(Y_t) + \text{Cov}(Y_{t+\tau}, X_t) + \mathbf{E}(Y_{t+\tau})\mathbf{E}(X_t) + 0 \\ &= R_{\mathcal{X}}(\tau) + 0 + 0 + 0 + 0 + 0 \end{aligned}$$

provided that $\tau > 0$. Thus, such a noise corruption of a WSS RP is also WSS.

Let \mathcal{X} be a WSS process entering a system T with impulse response h and transfer function H . Below, we show that the resulting output RP \mathcal{Y} is WSS

$$m_{\mathcal{Y}}(t) = \mathbf{E}\left(\int h(s)X_{t-s}ds\right) = \int h(s)\mathbf{E}(X_{t-s})ds = \mathbf{E}(X_0) \int h(s)ds = \mathbf{E}(X_0)H(0)$$

where we used the fact that the mean of a WSS RP is constant.

$$\begin{aligned} R_{\mathcal{Y}}(t, t + \tau) &= \mathbf{E}\left(\int h(s)X_{t-s}ds \cdot \int h(r)X_{t+\tau-r}dr\right) = \mathbf{E}\left(\iint h(s)h(r)X_{t-s}X_{t+\tau-r}dsdr\right) \\ &= \iint h(s)h(r)\mathbf{E}(X_{t-s}X_{t+\tau-r})dsdr = \iint h(s)h(r)R_{\mathcal{X}}(\tau + s - r)dsdr = g(\tau) \end{aligned}$$

where we used the linearity of expectation. Since $R_{\mathcal{Y}}(t, t + \tau)$ is a function of τ and the mean function is constant $\mathcal{Y} = T(\mathcal{X})$ is WSS.

Above, we characterizes the autocorrelation of the response \mathcal{Y} with the autocorrelation of the input \mathcal{X} and the impulse response of the system. In a similar way we can find a characterization of the power spectral density of the output in terms of the input the transfer function of the system

$$\begin{aligned} S_{\mathcal{Y}}(f) &= \int R_{\mathcal{Y}}(\tau)e^{-2\pi if\tau}d\tau = \int \left(\iint h(s)h(r)R_{\mathcal{X}}(\tau + s - r)dsdr\right) e^{-2\pi if\tau}d\tau \\ &= \iiint h(s)h(r)e^{-2\pi if(u+r-s)}R_{\mathcal{X}}(u)dsdrdu = \int h(s)e^{2\pi ifs}ds \int h(r)e^{-2\pi ifr}dr \int R_{\mathcal{X}}(u)e^{-2\pi ifu}du \\ &= \left(\int (h(s)e^{-2\pi ifs})^* ds\right) H(f)S_{\mathcal{X}}(f) = \left(\int h(s)e^{-2\pi ifs}ds\right)^* H(f)S_{\mathcal{X}}(f) = H(f)^*H(f)S_{\mathcal{X}}(f) = |H(f)|^2S_{\mathcal{X}}(f) \end{aligned}$$

where we used the change of variables $(s, r, \tau) \rightarrow (s, r, u)$ with $u = \tau + s - r$, $du = d\tau$, and y^* represents the complex conjugate of y .

The above relations characterizing the autocorrelation function or the power spectral density function of \mathcal{Y} are only partial descriptions of \mathcal{Y} . However, if \mathcal{X} is a Gaussian process, \mathcal{Y} is a Gaussian RP as well (linear transformation of a Gaussian RP results in a Gaussian RP) and it is entirely characterized by its mean and auto-correlation. We can therefore use the above relations to obtain the precise stochastic nature of an LTI-filtered Gaussian RP.