Functions of a Random Variable

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Since a RV X is a function $X : \Omega \to \mathbb{R}$, we can compose it with a function $g : \mathbb{R} \to \mathbb{R}$. The resulting function $g \circ X : \Omega \to \mathbb{R}$ is a RV denoted $g(X)$ i.e. $g(X)(\omega) = g(X(\omega))$. If $Y = g(X)$ is a RV that is a function of X, we can compute probabilities events relating Y through probabilities of events concerning X:

$$P(Y \in B) = P(\{\omega \in \Omega : Y(\omega) \in B\}) = P(\{\omega \in \Omega : g(X(\omega)) \in B\}) = P(\{\omega \in \Omega : X(\omega) \in g^{-1}(B)\}) \quad (1)$$
$$= P(X \in g^{-1}(B))$$

where $g^{-1}(B) = \{r \in \mathbb{R} : g(r) \in B\}$. In other words if we can compute quantities such as $P(X \in A)$ we can also compute quantities such as $P(g(X) \in B)$.

Discrete RV

For discrete RV X and $Y = g(X)$ equation (1) points the following relation between the two pmfs

$$p_Y(y) = P(Y = y) = P(g(X) = y) = P(X \in g^{-1}(\{y\})) = \sum_{x:g(x)=y} p_X(x).$$

In particular if g is one-to-one $p_Y(y) = p_X(x)$ for x such that $g(x) = y$.

For example, consider the RV X that counts the number of heads in three coin tosses. Recall that we have $p_X(0) = p_X(3) = 1/8, p_X(1) = p_X(2) = 3/8$ and $p_X(x) = 0$ otherwise. If $g(r) = r^2 - 2$, the pmf of the RV $Y = g(X)$ is $p_Y(-2) = p_X(0) = 1/8, p_Y(7) = p_X(3) = 1/8$, $p_Y(-1) = p_X(1) = 3/8$, $p_Y(2) = 3/8$ and 0 otherwise. If $g(r) = 0$ for $r < 1.5$ and $g(r) = 1$ for $r \geq 1.5$ the pmf of $Y = g(X)$ is $p_Y(0) = p_X(0) + p_X(1) = 1/2$ and $p_Y(1) = p_X(2) + p_X(3) = 1/2$.

Continuous RV

If X is a continuous RV, $Y = g(X)$ may be either a continuous or a discrete RV. For example, if X is a RV measuring the distance of a dart from the origin in a disc-shaped dart board with radius 1, X is continuous. Assume $g(r) = 0$ if $r < 0.1$ and $g(r) = 1$ otherwise. Then $g(X)$ is a discrete RV with the following pmf $p_Y(0) = P(Y = 0) = P(g(X) = 0) = \pi 0.1^2$, $p_Y(1) = 1 - p_Y(0)$ and $p_Y(y) = 0$ for all other y.

In the case $Y = g(X)$ is a continuous RV things get more interesting. There are three ways of computing $P(Y \in B)$ based on quantities like $P(X \in A)$. Recall that if Y is continuous we can compute $P(Y \in B)$ if we know the cdf or pdf of Y. Thus we can proceed in two ways: finding out the cdf of Y and then the pdf by differentiating, or finding out the pdf of Y directly.

To find the cdf of Y we can try to compute it directly by using equation (1)

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y).$$

For example, if X is the distance of the dart in the dart board example above, and $g(r) = r^2$, $Y = g(X) = X^2$ is a continuous RV with the cdf

$$F_Y(y) = P(X^2 \leq y) = P(\{\omega \in \Omega : X(\omega) \leq \sqrt{y}\}) = P(X \leq \sqrt{y}) = F_X(\sqrt{y}) = \pi \sqrt{y}^2 / \pi = y$$

for $0 < y < 1$, $F_Y(y) = 0$ for $y \leq 0$ and $F_Y(y) = 1$ for $y \geq 1$. We can then differentiate the cdf to find the pdf $f_Y(y) = 1$ for $0 < y < 1$ and 0 otherwise which enables us to compute quantities such as $P(Y \in B) = \int_B f_Y(y)dy$.

The second method computes the pdf of Y directly from the pdf of X, but only works if $g : \mathbb{R} \to \mathbb{R}$ is strictly monotone (strictly decreasing or strictly increasing) and differentiable on the range of X:

$$f_{g(X)}(y) = \frac{1}{|g'(g^{-1}(y))|} f_X(g^{-1}(y)).$$

This equation may be proved by the chain rule and elementary calculus. Assume that g is strictly increasing (the other case is similar). Then

$$\begin{aligned} f_Y(y) =& F_Y'(y) = \frac{d}{dy}P(g(X) \leq y) = \frac{d}{dy}P(X \leq g^{-1}(y)) = \frac{d}{dy}F_X(g^{-1}(y)) \\ &= F_X'(g^{-1}(y))\frac{d}{dy}g^{-1}(y) = F_X'(g^{-1}(y))\frac{1}{g'(g^{-1}(y))} = F_X'(g^{-1}(y))\frac{1}{|g'(g^{-1}(y))|} \end{aligned}$$

where we used the result that the derivative of an increasing function is positive and $(d/dy)g^{-1}(y) = 1/g'(g^{-1}(y))$.

Returning to the dart board example, the RV X has cdf $F_X(x) = x^2$ for $0 < x < 1$ and hence $f_X(x) = 2x$ in that range as well. The function $g(r) = r^2$ is increasing and differentiable in the range of X (which is $(0, 1)$) and therefore the RV $Y = g(X) = X^2$ has the pdf

$$f_Y(y) = \frac{1}{|g'(\sqrt{y})|} f_X(\sqrt{y}) = \frac{1}{2\sqrt{y}} 2\sqrt{y} = 1$$

for $0 < y < 1$ and 0 otherwise which agrees with the computation of F_Y and f_Y via the cdf method.

The third method is based on the moment generating function and will be described in a separate handout on that topic.