Markov Chains

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In this note we summarize the basic definitions and properties involving Markov chains. For more details and proofs see [1]. A Markov chain is a discrete-time discrete-state stochastic process that satisfies the Markov assumption $p(X_i|X_{i-1},\ldots,X_{i-k}) = p(X_i|X_{i-1})$ for all $1 \leq k < i = 1,2,\ldots$. By discrete time we mean that the process is a distribution over $\{X_i : i \in \mathbb{N}\}$ ($\mathbb{N}$ are the non-negative integers). By discrete space we mean $X_i \in \mathbb{N}$. Because of the Markov assumption the process is characterized by an (infinite) transition matrix $T_{ij}(t) = p(X_{t+1} = j|X_t = i)$ and by a (infinite) vector of initial state probabilities $\rho_i = p(X_0 = i)$. In this note we will assume (as is usually done) that the chain is homogeneous i.e., the transition matrix is independent of the time: $T(t) = T$ for all t. As a consequence we have that[1] $[T^2]_{ij} = p(X_{t+2} = j|X_t = i)$ and more generally T^n is the n-step transition matrix whose rows are non-negative numbers summing to one. Similarly using matrix notation we have $p(X_n) = \rho^\top T^n$. Note that when there are only a finite number of states, T is a finite matrix, and ρ is a finite dimensional vector.

A state i is said to be *accessible* from state j if for some integer $n > 0$, $T^n_{ij} > 0$ i.e., there is some positive probability of moving from i to j after a finite period of time. If i is accessible from j and j is accessible from i the two state are said to *communicate* with each other, denoted by $i \leftrightarrow j$. It can be shown that $\leftrightarrow$ relation induces a partition into equivalence classes of communicating states. If there is only one equivalence class the chain is said to be *irreducible*. The *period* $d(i)$ of a state i is the greatest common divisor of all integer $n \geq 1$ for which $T^n_{ii} > 0$. It can be shown that all states within the same equivalence class of the $\leftrightarrow$ relation have the same period. If the period of a state is one we say that it (and its equivalence class) is aperiodic.

A state i is *recurrent* if starting from i the probability of returning to i after some finite time is 1. A non-recurrent state is said to be transient. It can be shown that a state i is recurrent iff assuming $X_0 = i$ the expected number of returns to i is infinity $\sum_{n=1}^{\infty} T^n_{ii} = \infty$. Furthermore, it can be shown that all states within an equivalence class of communicating states are either all recurrent or all transient. For example a chain with a positive T is irreducible, aperiodic and recurrent. If $\pi_i \stackrel{\text{def}}{=} \lim_n T^n_{ii} > 0$ for some i in a recurrent class then the same holds for the entire class and we call the state (and the class) *positive recurrent* or *strongly ergodic* (otherwise we say that it is *null-recurrent* or *weakly ergodic*). If the state space is finite all the states are either positive recurrent or transient (there are no null-recurrent states).

Proposition 1 (Basic Markov Chain Theorem)**.** *In an irreducible positive-recurrent aperiodic chain the limit π exists and is an eigenvector of $T^\top$ with eigenvalue 1 i.e., it is characterized by the following equations*

$$\pi_i \geq 0, \quad \sum_{i=0}^{\infty} \pi_i = 1, \quad T^\top \pi = \pi.$$

It j is a transient state then $T^n_{ij} \to 0$ regardless of i. If i,j are in the same aperiodic recurrent class $T^n_{ij} \to \pi_j \geq 0$. If i,j are in the same periodic recurrent class the same holds if we replace T^n_{ij} by $n^{-1}\sum_{m=1}^{n} T^m_{ij}$. If $j \in C$ for an aperiodic recurrent class C and i is in a transient class R we have

$$\lim_{n\to\infty} T^n_{ij} = \pi_i(C) \lim_{n\to\infty} T^n_{jj} = \pi_i(C)\pi_j$$

where $\pi_i(C)$ is the probability of arriving at the recurrent class C if the initial state is $i \in R$. The probabilities $\pi_i(C)$ may be found by solving the equations (where $\pi^n_i(C)$ below is the probability of arriving at recurrent

[1]For notational brevity we use matrix notation to multiply two infinite matrices or an infinite matrix with an infinite vector with the obvious interpretation i.e., $[T^2]_{ij} = \sum_{k=0}^{\infty} T_{ik}T_{kj}$.

class C if the initial state is i for the first time after n steps)

$$\pi_i(C) = \pi_i^1(C) + \sum_{n=2}^{\infty} \pi_i^n(C) = \pi_i^1(C) + \sum_{j\in R} T_{ij} \sum_{n=2}^{\infty} \pi_j^{n-1}(C) = \pi_i^1(C) + \sum_{j\in R} T_{ij}\pi_j(C), \quad i \in R. \tag{1}$$

Note in particular that in an irreducible positive-recurrent aperiodic chain the limiting distribution $\lim_{n\to\infty} T_{ij}^n = \pi_j$ does not depend on i. Consequentially, the process will converge as $n \to \infty$ to a stationary state distribution π regardless of the initial state or the initial state distribution $\rho = p(X_0)$.

Analogous to the $\pi_i(C)$ above are $\tau_i^n(R)$, defined as the probabilities that starting from state $i \in R$ the process remains in the transient class R for the next n transitions. We show by induction that $\tau_i^n(R)$ is decreasing in n: $\tau_i^2(R) = \sum_{j\in R} T_{ij}\tau_j^1(R) \leq \sum_{j\in R} T_{ij} = \tau_i^1(R)$, and assuming the induction hypothesis holds for $\tau_i^n(R) \leq \tau_i^{n-1}(R)$, $i \in T$, we have $0 \leq \tau_i^{n+1} = \sum_{j\in R} T_{ij}\tau_j^n \leq \sum_{j\in R} T_{ij}\tau_j^{n-1} = \tau_i^n$. Since $\tau_i^n(R)$ is a non-negative decreasing sequence in n it converges to a limit $\tau_i(R)$ corresponding to the probabilities of never enterring a recurrent class (after being in state i) which satisfies

$$\tau_i(R) = \sum_{j\in R} T_{ij}\tau_j(R), \quad i \in R. \tag{2}$$

If the only bounded solution for (2) is $\tau(R) \equiv 0$ then with probability 1 starting from any transient state in R leads to entrance into the recurrent class. The following result is useful in determining whether a given chain is recurrent or transient.

Proposition 2. *An irreducible Markov chain is transient iff $[Ty]_i = y_i, i > 0$ has a bounded non-constant solution y. If there exists a solution y for $[Ty]_i \leq y_i, i > 0$ with $y_i \to \infty$ the chain is recurrent.*

Infinite Gambler's Ruin Example

The gambler's ruin process models a gambler playing an infinitely rich opponent (such as a casino): $T_{0*} = (1, 0, 0, \ldots)$, $T_{1*} = (q, 0, p, 0, 0, \ldots)$, $T_{2*} = (0, q, 0, p, 0, 0, \ldots)$. In each turn the gambler wins one money unit with probability p and loses the same amount with probability q. The index of the state reflects the gambler's money with the zero state being an absorbing state (where the gambler cannot gamble any more) also known as the gambler's ruin. The equivalence classes of communicable states are $C = \{0\}$ (period 0, recurrent) and $R = \{1, 2, \ldots, \}$ (period 2, transient). The system of equations (1) for $u_j = \pi_j(\{0\})$ become

$$u_1 = q + pu_2, \quad u_i = qu_{i-1} + pu_{i+1}, \quad i \geq 2$$

whose solution for u (assuming it is bounded as it is a vector of probability values) is $u_i \equiv 1$ if $q \geq p$ and $u_i = (q/p)^i$ otherwise. The remarkable conclusion is that when the gamble is stacked against the gambler or *even if it is a fair gamble* i.e., $q \geq p$, the gambler will be ruined with probability 1 *regardless of their initial fortune*. Moreover, even if the gambler has favorable odds $p > q$ he or she will still be ruined with probability $(q/p)^{\text{initial captial}}$. The lack of symmetry between the gambler and the casino is due to the fact that the casino has unlimited funds. It is indeed not recommended to play against such an opponent indefinitely.

References

[1] S. Karlin and H. M. Taylor. *A First Course in Stochastic Processes*. Academic Press, second edition, 1975.