

The Method of Moment Estimator

Guy Lebanon

May 12, 2006

We have defined some desirable properties of estimators such as efficiency, consistency and sufficiency. However, we have not seen any general purpose method for obtaining good estimators. The method of moment estimator and maximum likelihood estimator are two such general purpose methods. They generally obtain consistent estimators and are usually straightforward to calculate using a computer. In this note we present the method of moment estimation (mome) method.

Definition 1. *The k -moment of a RV X is $E(X^k)$.*

The motivation behind the mome is that if we have a good estimator $\hat{\theta}$, the distribution that underlies $\hat{\theta}$ should be similar to the distribution of θ - where similarity is compared by equality of moments. However, we do not know the moments of the distribution that corresponds to θ since we don't know the value of θ . For this reason we approximate it by the sample moment - the moment computed by the given sample (that was generated from θ). In other words, we would choose $\hat{\theta}$ such that

$$E_{\hat{\theta}}(X) = \frac{1}{n} \sum_{i=1}^n X_i.$$

In case that θ is a vector of k components, we need more than one equation. Specifically we have k -unknown parameters so we need k equations. Therefore we require the equality of the first k moments.

Definition 2. *Let X_1, \dots, X_n be iid sample from P - a distribution with a k -dimensional parameter vector θ . The method of moment estimator (mome) $\hat{\theta}$ is the solution to the following system of equations*

$$E_{\hat{\theta}}(X^j) = \frac{1}{n} \sum_{i=1}^n X_i^j \quad k = 1, \dots, k.$$

Example: Let $X_1, \dots, X_n \sim U(0, \theta)$. Since $E(X) = \theta/2$ the mome is the solution to

$$\hat{\theta}/2 = E_{\hat{\theta}}(X) = \frac{1}{n} \sum_{i=1}^n X_i = \bar{X}$$

which is $\hat{\theta} = 2\bar{X}$. This estimator can be shown to be consistent since it is unbiased (we showed this in an earlier note) and its variance goes to 0 in the limit $n \rightarrow \infty$.

Example: $X_1, \dots, X_n \sim \text{Gamma}(\alpha, \beta)$. We have $\theta = (\alpha, \beta)$ and so we will have to solve two equations. It is well known that $E(X) = \alpha\beta$ and $\text{Var}(X^2) = \alpha\beta^2$ whence $E(X^2) = \text{Var}(X) + E(X)^2 = \alpha\beta^2 + \alpha^2\beta^2$. The two equations are thus

$$\hat{\alpha}\hat{\beta} = \bar{X} \quad \hat{\alpha}\hat{\beta}^2 - \hat{\alpha}^2\hat{\beta}^2 = \frac{1}{n} \sum X_i^2.$$

Solving the first $\hat{\beta} = \bar{X}/\hat{\alpha}$ and substituting in the second we obtain $\hat{\alpha} = \frac{\bar{X}^2}{\frac{1}{n} \sum X_i^2 - \bar{X}^2}$ and $\hat{\beta} = \bar{X}/\hat{\alpha} = \frac{\frac{1}{n} \sum X_i^2 - \bar{X}^2}{\bar{X}}$.