The Poisson Process

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In this note we summarize the basic definitions and properties involving the Poisson process. For more details see [1]. The Poisson process is a continuous time ($t \geq 0$) discrete space $X_t \in \mathbb{N} = \{0, 1, \ldots, \}$ Markov process that follows from assuming that the probability of getting a single event in a small time interval $[t, t + \Delta]$ is $\Delta\lambda$ (for some $\lambda > 0$) and the probability of getting more than a single event in a small time interval is negligible. It is often used to model counting or accumulation of independent events, i.e., X_t is the number of cars arriving in an intersection or the number of calls arriving at a switchboard by time t.

Postulates and Differential Equation

It is useful to derive the Poisson process from first principles. It exposes its underlying assumptions and points the way for potential generalizations. The basic postulates are

1. The process is a Markov process with stationary transition probabilities i.e., $P_{ij}(t) = P(X_{t+u} = j | X_u = i)$, $t > 0$ is independent of $u \geq 0$.
2. $P(X_{t+h} - X_t = 1 | X_t = x) = \lambda h + o(h)$ i.e., $\lim_{h \to 0} P(X_{t+h} - X_t = 1 | X_t = x)/h = \lambda$
3. $X_0 = 0$ with probability 1
4. $P(X_{t+h} - X_t = 0 | X_t = x) = 1 - \lambda h + o(h)$.

Postulates 2 and 4 imply that the probability of multiple events occuring in a short time interval is negligible. Denoting $P_m(t) = P(X_t = m)$ the above postulates imply $P_0(t + h) = P_0(t)P_0(h) = P_0(t)(1 - \lambda h + o(h))$ which in turn imply $(P_0(t+h) - P_0(t))/h = -P_0(t)(\lambda h + o(h))/h = -P_0(t)(\lambda + o(1))$ and consequentially we get the differential equation $P_0'(t) = -aP_0(t)$ whose well known solution (subject to initial condition $X_0 = 0$ is $P_0(t) = e^{-\lambda t}$. From the law of total probability we get the following differential equation

$$\begin{aligned}
P_m(t + h) - P_m(t) &= P_m(t)P_0(h) + P_{m-1}(t)P_1(h) + \sum_{j=2}^{m} P_{m-j}(t)P_j(h) - P_m(t) \\
&= P_m(t)(1 - \lambda h + o(h)) + P_{m-1}(t)(\lambda h + o(h))) + o(h)C - P_m(t) \\
&= -\lambda P_m(t)h + \lambda P_{m-1}(t)h + o(h) \\
\Rightarrow \quad P_m'(t) &= \lim_{h \to 0} \frac{P_m(t + h) - P_m(t)}{h} = -\lambda P_m(t) + \lambda P_{m-1}(t)
\end{aligned}$$

Poisson Distribution

Solving the above differential equation recursively for $P_m(t)$ using $P_0(t) = e^{-\lambda t}$ and $P_m(0) = 0$ yields the solution $P_m(t) = \frac{\lambda^m t^m}{m!} \exp(-\lambda t)$ i.e., the marginal distribution of the process is Poisson $X_t \sim \text{Poisson}(\lambda t)$ with expectation and variance λt (see [1] for a proof).

Exponential Distribution

Denoting S_i as the times event i occurs, we have that the wait times $T_i = S_{i+1} - S_i$ between consecutive event occurrences are independent exponential random variables $\exp(\lambda)$. First, note that since $P_0(t) = e^{-\lambda t}$,

$P(\text{time of first occurrence} \leq z) = 1 - P(X_z = 0) = 1 - e^{-\lambda z}$ i.e., the time of first occurrence has an exponential distribution with parameter λ. The same holds if we start measuring time from a different time point $t > 0$ i.e., the time measured from any arbitrary $t \geq 0$ until the next event is distributed $\exp(\lambda)$ as well. Finally, selecting t as the time of previous event we have that the wait times between consecutive events are independent $\exp(\lambda)$ random variables.

Uniform Distribution

Given $X_t = n$, the distribution of the event occurrence times $S_1, \ldots, S_n$ are distributed as the order statistics of n uniform distribution RVs on $[0, t]$ (see [1] for a full proof)(see [1] for a proof).

$$P(S_1 \leq s_1, \ldots, S_n \leq s_n | X_t = n) = \frac{P(S_1 \leq s_1, \ldots, S_n \leq s_n | X_t = n)}{P(X_t = n)} = \frac{\lambda^n e^{-\lambda t}}{\lambda^n t^n e^{-\lambda t}/n!} \int_0^s \int_{u_1}^{s_1} \cdots \int_{u_{n-1}}^{s_n} du_n \cdots du_1$$

In particular, the distribution of the event time S_1 conditioned on $X_t = 1$ is $U[0, t]$.

Binomial Distribution

Examining the characterstic function (CF) of the Poisson distribution $\exp(\lambda(e^{it} - 1))$ it is easy to see that a sum of independent Poisson distribution with parameters $\lambda_j, j = 1, \ldots, k$ is a Poisson distribution with parameter $\sum_{j=1}^k \lambda_j$ (CF of a sum of RVs is the product of the individual CFs). Assuming two Poisson processes with parameters λ_1, λ_2 we have the following binomial distribution

$$\begin{aligned} P(X_t^{(1)} = k | X_t^{(1)} + X_t^{(2)} = n) &= \frac{P(X_t^{(1)} = k) P(X_t^{(2)} = n - k)}{P(X_t^{(1)} + X_t^{(2)} = n)} = \frac{\text{Poisson}(k\,; \lambda_1 t) \text{Poisson}(n - k\,; \lambda_2 t)}{\text{Poisson}(n\,; \lambda_1 t + \lambda_2 t)} \\ &= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-k} = \text{Bin}\left(k\,; n, \frac{\lambda_1}{\lambda_1 + \lambda_2}\right). \end{aligned}$$

Another case where the binomial distribution arises is as follows (assume below $u < t$ and $k < n$)

$$\begin{aligned} P(X_u = k | X_t = n) &= \frac{P(X_u = k) P(X_t - X_u = n - k)}{P(X_t = n)} = \frac{\text{Pois}(k\,; \lambda u) \text{Pois}(n - k\,; \lambda(t - u))}{\text{Pois}(n, ; \lambda t)} \\ &= \binom{n}{k} \frac{u^k (t - u)^{n-k}}{t^n} = \text{Bin}\left(k\,; n, \frac{u}{t}\right) \end{aligned}$$

References

[1] S. Karlin and H. M. Taylor. *A First Course in Stochastic Processes.* Academic Press, second edition, 1975.