Sample Space, Events and Probabilities

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Definition 1. *A sample space Ω associated with a random experiment is the set of all possible outcomes in it. A sample space can be finite, countable-infinite, or uncountable infinite.*

Definition 2. *A event E in a sample space Ω associated with a random experiment is a subset of Ω thus: $E \subset \Omega$. The sets $\emptyset$ and Ω are always events, but other subsets of Ω may or may not be events.*

For example in the random experiment of tossing a coin three times and observing the results (heads or tails) with ordering the sample space is

$$\Omega = \{\text{HHH,HHT,HTH,HTT,THH,THT,TTH,TTT}\}$$

and the event

$$E = \{\text{HHH,HHT,HTT,HTH}\} \subset \Omega$$

describes "a head was obtained in the first coin toss". Note that the sample space here is finite.

An example of continuous sample space is the following. Suppose a dart is thrown at a round board. The sample space contains the location of the dart if it lands in the board and if it lands outside the board - it does not matter where precisely (only that it is not on the board). The (infinite) sample space is

$$\Omega = \{(x, y) : x, y \in \mathbb{R}, x^2 + y^2 < 1\} \cup \{\text{outside}\}$$

and an event describing a bulls-eye hit is

$$E = \{(x, y) : x, y \in \mathbb{R}, x^2 + y^2 < 0.1\} \subset \Omega.$$

For an event E, the outcome of the random experiments $\omega \in \Omega$ is either in E: $\omega \in E$ or not $\omega \notin E$. In the first case we say that E occurred and in the second case we say that E did not occur. Set operation may be given intuitive description if the sets are events. $A \cup B$ is the event of either A or B occurring and$A \cap B$ is the event of both A and B occurring. A^c (in the complement, the universal set is taken to be Ω) is the event not-A and so on. If the events A, B are disjoint $A \cap B = \emptyset$ it means that the two events can not happen at the same time (since no outcome of the random experiment $\omega \in \Omega$ may be belong to both A, B).

Definition 3. *Let Ω be a sample space associated with a random experiment. A probability measure or distribution is a real-valued function P on events $E \subset \Omega$ that satisfies the following axioms:*

1. $P(E) \geq 0 \quad \forall E$
2. $P(\Omega) = 1$
3. *If $\{E_i\}_{i=1}^{\infty}$ is a sequence of mutually disjoint events, then*

$$P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i).$$

The third axiom implies also finite additivity: For every finite sequence of mutually disjoint events $\{E_i\}_{i=1}^{n}$ we have

$$P(E_1 \cup \cdots \cup E_n) = P(E_1) + \cdots + P(E_n).$$

The axioms may be used to to derive the following properties of probability:

Theorem 1. $P(A^c) = 1 - P(A)$

Proof.

$$1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c)$$

□

Theorem 2. $P(A) \leq 1$

Proof. By the previous result, $P(A) = 1 - P(A^c)$. Since all probabilities are non-negative $1 - P(A^c)$ is less than or equal to 1. □

Theorem 3. $P(\emptyset) = 0$

Proof.

$$1 = P(\Omega) = P(\Omega \cup \emptyset) = P(\Omega) + P(\emptyset) = 1 + P(\emptyset)$$

□

Theorem 4. *If* $A \subset B$, $P(B) = P(A) + P(B \setminus A)$. *It also follows then that* $P(B) \geq P(A)$.

Proof.

$$P(B) = P(A \cup (B \setminus A)) = P(A) + P(B \setminus A)$$

□

Theorem 5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof.

$$\begin{aligned} P(A \cup B) &= P((A \setminus (A \cap B)) \cup (B \setminus (A \cap B))) \cup (A \cap B)) = P((A \setminus (A \cap B)) + P(B \setminus (A \cap B))) + P(A \cap B) \\ &= P(A) - P(A \cap B) + P(B) - P(A \cap B) + P(A \cap B) = P(A) + P(B) - P(A \cap B) \end{aligned}$$

□