Stationary and Wide Sense Stationary Processes

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Definition 1. *A RP* $\mathcal{X} = \{X_t : t \in \mathbb{R}\}$ *or* $\mathcal{X} = \{X_n : n = \ldots, -2, -1, 0, 1, 2, \ldots\}$ *is stationary if all its* k*-order marginals do not depend on a translation by a constant: for any* $k > 0$ *and for any time indices* $t_1, \ldots, t_k$,

$$\forall \tau \in \mathbb{R} \quad f_{X_{t_1},\ldots,X_{t_k}}(x_1, \ldots, x_k) = f_{X_{t_1+\tau},\ldots,X_{t_k+\tau}}(x_1, \ldots, x_k).$$

The above expression is for continuous-valued RP. If the RP is discrete valued, replace the pdfs with pmfs.

Definition 2. *A RP* $\mathcal{X} = \{X_t : t \in \mathbb{R}\}$ *or* $\mathcal{X} = \{X_n : n = \ldots, -2, -1, 0, 1, 2, \ldots\}$ *is wide sense stationary (WSS) if its mean function* $m_{\mathcal{X}}(t)$ *is constant and its autocorrelation* $R_{\mathcal{X}}(t, s)$ *is a function of* $|s-t|$*. In this case, we abuse notation (by wide convention) and consider the autocorrelation as a function of one variable - the difference between the two time points* $R_{\mathcal{X}}(\tau) = R_{\mathcal{X}}(t, t+\tau)$.

A stationary RP is also WSS but the converse is not true in general. WSS processes will prove to be useful for signal processing purposes as they simplify some expressions in a considerable manner. Note that the autocovariance of a WSS RP is also a function of the difference between the two time points.

Below are some properties of the autocorrelation function of WSS RP $\mathcal{X}$:

1. $R_{\mathcal{X}}(0) = \mathsf{E}(X_t^2)$ for all t (it measures the average "power" of the process which is fixed for all t). Since X_t^2 is a non-negative RV, $R_{\mathcal{X}}(0) \geq 0$.

2. $R_{\mathcal{X}}$ is an even function
$$R_{\mathcal{X}}(\tau) = \mathsf{E}(X_t X_{t+\tau}) = \mathsf{E}(X_{t+\tau} X_t) = R_{\mathcal{X}}(-\tau).$$

3. The probability of a change in the RP realization after some time is bounded by the difference in the autocorrelation function:
$$P(|X_{t+\tau} - X_t| \geq \epsilon) = P((X_{t+\tau} - X_t)^2 \geq \epsilon^2) \leq \frac{\mathsf{E}((X_{t+\tau} - X_t)^2)}{\epsilon^2}$$
$$= \frac{\mathsf{E}((X_{t+\tau}^2 + X_t^2 - 2X_{t+\tau}X_t))}{\epsilon^2} = \frac{2(R_{\mathcal{X}}(0) - R_{\mathcal{X}}(\tau))}{\epsilon^2}$$
where the inequality follows from Markov inequality.

4. The autocorrelation function attains its maximum at $\tau = 0$.

   First, we show that since the expectation of a non-negative RV is non-negative:
$$0 \leq \mathsf{E}((X/\sqrt{\mathsf{E}(X^2)} - Y/\sqrt{\mathsf{E}(Y^2)})^2) = \mathsf{E}(X^2/\mathsf{E}(X^2) + Y^2/\mathsf{E}(Y^2) - 2XY/\sqrt{\mathsf{E}(X^2)\mathsf{E}(Y^2)})$$
$$= 1 + 1 - 2\mathsf{E}(XY)/\sqrt{\mathsf{E}(X^2)\mathsf{E}(Y^2)} = 2(1 - \mathsf{E}(XY)/\sqrt{\mathsf{E}(X^2)\mathsf{E}(Y^2)}) \Rightarrow \mathsf{E}(XY) \leq \sqrt{\mathsf{E}(X^2)\mathsf{E}(Y^2)}$$
   and hence $(\mathsf{E}(XY))^2 \leq \mathsf{E}(X^2)\mathsf{E}(Y^2)$. Using this last result we have
$$(R_{\mathcal{X}}(\tau))^2 = (\mathsf{E}(X_{t+\tau}X_t))^2 \leq \mathsf{E}((X_{t+\tau})^2)\mathsf{E}((X_t))^2 = (R_{\mathcal{X}}(0))^2.$$
   The above result, together with the fact that $R_{\mathcal{X}}(0) = \mathsf{E}(X_t^2)$ is non-negative (property 1) yields the required result: $R_{\mathcal{X}}(0) \geq R_{\mathcal{X}}(\tau)$ for all τ.