Appendix A

Set Theory

This chapter describes set theory, a mathematical theory that underlies all of modern mathematics.

A.1 Basic Definitions

Definition A.1.1. A set is an unordered collection of elements.

Sets may be described by listing their elements between curly braces, for example $\{1, 2, 3\}$ is the set containing the elements 1, 2, and 3. Alternatively, we an describe a set by specifying a certain condition whose elements satisfy, for example $\{x : x^2 = 1\}$ is the set containing the elements 1 and -1 (assuming x is a real number).

We make the following observations.

- There is no importance to the order in which the elements of a set appear. Thus $\{1, 2, 3\}$, is the same set as $\{3, 2, 1\}$.
- An element may either appear in a set or not, but it may not appear more than one time.
- Sets are typically denoted by an uppercase letter, for example A or B.
- It is possible that the elements of a set are sets themselves, for example $\{1, 2, \{3, 4\}\}$ is a set containing three elements (two scalars and one set). We typically denote such sets with calligraphic notation, for example $\mathcal{U}$.

Definition A.1.2. If a is an element in a set A, we write $a \in A$. If a is not an element of A, we write $a \notin A$. The empty set, denoted by $\emptyset$ or $\{\}$, does not contain any element.

Definition A.1.3. A set A with a finite number of elements is called a finite set and its size (number of elements) is denoted by $|A|$. A set with an infinite number of elements is called an infinite set.

Definition A.1.4. We denote $A \subset B$ if all elements in A are also in B. We denote $A = B$ if $A \subset B$ and $B \subset A$, implying that the two sets are identical. The difference between two sets $A \setminus B$ is the set of elements in A but not in B. The complement of a set A with respect to a set Ω is $A^c = \Omega \setminus A$ (we may omit the set Ω if it is obvious from context). The symmetric difference between two sets A, B is

$$A \triangle B = \{x : x \in A \setminus B \text{ or } x \in B \setminus A\}.$$

Example A.1.1. *We have* $\{1,2,3\} \setminus \{3,4\} = \{1,2\}$ *and* $\{1,2,3\} \triangle \{3,4\} = \{1,2,4\}$. *Assuming* $\Omega = \{1,2,3,4,5\}$, *we have* $\{1,2,3\}^c = \{4,5\}$.

In many cases we consider multiple sets indexed by a finite or infinite set. For example $U_\alpha, \alpha \in A$ represents multiple sets, one set for each element of A.

Example A.1.2. *Below are three examples of multiple sets,* $U_\alpha, \alpha \in A$. *The first example shows two sets:* $\{1\}$ *and* $\{2\}$. *The second example shows multiple sets:* $\{1,-1\}$, $\{2,-2\}$, $\{3,-3\}$, *and so on. The third example shows multiple sets, each containing all real numbers between two consecutive natural numbers.*

$$\begin{aligned} U_i &= \{i\}, & i &\in A = \{1,2\}, \\ U_i &= \{i, -i\}, & i &\in A = \mathbb{N} = \{1,2,3,\ldots\}, \\ U_\alpha &= \{\alpha + r : 0 \leq r \leq 1\}, & \alpha &\in A = \mathbb{N} = \{1,2,3,\ldots\}. \end{aligned}$$

Definition A.1.5. For multiple sets $U_\alpha, \alpha \in A$ we define the union and intersection operations as follows:

$$\bigcup_{\alpha \in A} U_\alpha = \{u : u \in U_\alpha \text{ for one or more } \alpha \in A\}$$
$$\bigcap_{\alpha \in A} U_\alpha = \{u : u \in U_\alpha \text{ for all } \alpha \in A\}.$$

Figure A.1 illustrates these concepts in the case of two sets A, B with a non-empty intersection.

Definition A.1.6. The sets $U_\alpha, \alpha \in A$ are disjoint or mutually disjoint if $\cap_{\alpha \in A} U_\alpha = \emptyset$ and are pairwise disjoint if $\alpha \neq \beta$ implies $U_\alpha \cap U_\beta = \emptyset$. A union of pairwise disjoint sets U_α, $\alpha \in A$ is denoted by $\uplus_{\alpha \in A} U_\alpha$.

Example A.1.3.

$$\{a,b,c\} \cap \{c,d,e\} = \{c\}$$
$$\{a,b,c\} \cup \{c,d,e\} = \{a,b,c,d,e\}$$
$$\{a,b,c\} \setminus \{c,d,e\} = \{a,b\}.$$

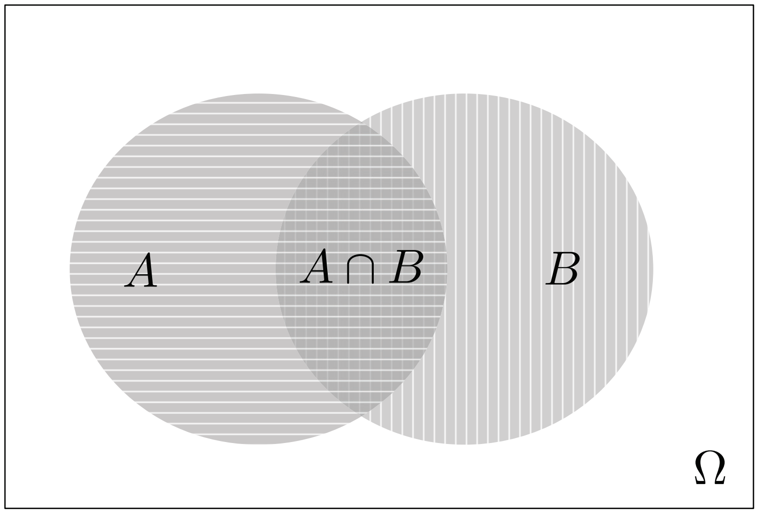

Figure A.1: Two circular sets A, B, their intersection $A \cap B$ (gray area with horizontal and vertical lines), and their union $A \cup B$ (gray area with either horizontal or vertical lines or both). The set $\Omega \setminus (A \cup B) = (A \cup B)^c = A^c \cap B^c$ is represented by white color.

Example A.1.4. *If* $A_1 = \{1\}, A_2 = \{1, 2\}, A_3 = \{1, 2, 3\}$ *we have*

$$\{A_i : i \in \{1, 2, 3\}\} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$$
$$\bigcup_{i \in \{1,2,3\}} A_i = \{1, 2, 3\}$$
$$\bigcap_{i \in \{1,2,3\}} A_i = \{1\}.$$

The properties below are direct consequences of the definitions above.

Proposition A.1.1. *For all sets* $A, B, C \subset \Omega$*,*

1. *Union and intersection are commutative and distributive:*

$$A \cup B = B \cup A, \qquad (A \cup B) \cup C = A \cup (B \cup C)$$
$$A \cap B = B \cap A, \qquad (A \cap B) \cap C = A \cap (B \cap C)$$

2. $(A^c)^c = A, \qquad \emptyset^c = \Omega, \qquad \Omega^c = \emptyset$

3. $\emptyset \subset A$

4. $A \subset A$

5. $A \subset B$ *and* $B \subset C$ *implies* $A \subset C$

6. *$A \subset B$ if and only if $B^c \subset A^c$*

7. *$A \cup A = A = A \cap A$*

8. *$A \cup \Omega = \Omega, \qquad A \cap \Omega = A$*

9. *$A \cup \emptyset = A, \qquad A \cap \emptyset = \emptyset.$*

Definition A.1.7. The power set of a set A is the set of all subsets of A, including the empty set $\emptyset$ and A. It is denoted by 2^A.

Proposition A.1.2. *If A is a finite set then*

$$|2^A| = 2^{|A|}.$$

Proof. We can describe each element of 2^A by a list of $|A|$ 0 or 1 digits (1 if the corresponding element is selected and 0 otherwise). The proposition follows since there are $2^{|A|}$ such lists (see Proposition 1.6.1). ∎

Example A.1.5.

$$2^{\{a,b\}} = \{\emptyset, \{a,b\}, \{a\}, \{b\}\}$$
$$|2^{\{a,b\}}| = 4$$
$$\sum_{A \in 2^{\{a,b\}}} |A| = 0 + 2 + 1 + 1 = 4.$$

The R package `sets` is convenient for illustrating basic concepts.

```
library(sets)
A = set("a", "b", "c")
2^A

## {{}, {"a"}, {"b"}, {"c"}, {"a", "b"},
##  {"a", "c"}, {"b", "c"}, {"a", "b",
##  "c"}}

A = set("a", "b", set("a", "b"))
2^A

## {{}, {"a"}, {"b"}, {{"a", "b"}}, {"a",
##  "b"}, {"a", {"a", "b"}}, {"b", {"a",
##  "b"}}, {"a", "b", {"a", "b"}}}

A = set(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)
length(2^A)  # = 2^10

## [1] 1024
```

Proposition A.1.3 (Distributive Law of Sets)**.**

$$\left(\bigcup_{\alpha\in Q} A_\alpha\right)\bigcap C = \bigcup_{\alpha\in Q}\left(A_\alpha \bigcap C\right)$$
$$\left(\bigcap_{\alpha\in Q} A_\alpha\right)\bigcup C = \bigcap_{\alpha\in Q}\left(A_\alpha \bigcup C\right).$$

Proof. We prove the first statement for the case of $|Q| = 2$. The proof of the second statement is similar, and the general case is a straightforward extension.

We prove the set equality $U = V$ by showing $U \subset V$ and $V \subset U$. Let x belong to the set $(A \cup B) \cap C$. This means that x is in C and also in either A or B, which implies $x \in (A \cap C) \cup (B \cap C)$. On the other hand, if x belongs to $(A \cap C) \cup (B \cap C)$, x is in $A \cap C$ or in $B \cap C$. Therefore $x \in C$ and also x is in either A or B, implying that $x \in (A \cup B) \cap C$. ∎

Proposition A.1.4.

$$S \setminus \bigcup_{\alpha\in Q} A_\alpha = \bigcap_{\alpha\in Q}(S \setminus A_\alpha)$$
$$S \setminus \bigcap_{\alpha\in Q} A_\alpha = \bigcup_{\alpha\in Q}(S \setminus A_\alpha).$$

Proof. We prove the first result. The proof of the second result is similar.

Let $x \in S \setminus \cup_{\alpha\in Q} A_\alpha$. This implies that x is in S but not in any of the A_α sets, which implies $x \in S \setminus A_\alpha$ for all α, and therefore $x \in \cap_{\alpha\in Q}(S \setminus A_\alpha)$. On the other hand, if $x \in \cap_{\alpha\in Q}(S \setminus A_\alpha)$, then x is in each of the $S \setminus A_\alpha$ sets. Therefore x is in S but not in any of the A_α sets, hence $x \in S \setminus \cup_{\alpha\in Q} A_\alpha$. ∎

Corollary A.1.1 (De-Morgan's Law)**.**

$$\left(\bigcup_{\alpha\in Q} A_\alpha\right)^c = \bigcap_{\alpha\in Q} A_\alpha^c$$
$$\left(\bigcap_{\alpha\in Q} A_\alpha\right)^c = \bigcup_{\alpha\in Q} A_\alpha^c.$$

Proof. This is a direct corollary of Proposition A.1.4 when $S = \Omega$. ∎

Definition A.1.8. We define the sets of all natural numbers, integers, and rational numbers as follows:

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$
$$\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$$
$$\mathbb{Q} = \{p/q : p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\}\}.$$

The set of real numbers[1] is denoted by $\mathbb{R}$.

Definition A.1.9. We denote the closed interval, the open interval and the two types of half-open intervals between $a, b \in \mathbb{R}$ as

$$\begin{aligned}[a,b] &= \{x \in \mathbb{R} : a \leq x \leq b\} \\ (a,b) &= \{x \in \mathbb{R} : a < x < b\} \\ [a,b) &= \{x \in \mathbb{R} : a \leq x < b\} \\ (a,b] &= \{x \in \mathbb{R} : a < x \leq b\}.\end{aligned}$$

Example A.1.6.

$$\begin{aligned}\bigcup_{n\in\mathbb{N}} [0, n/(n+1)) &= [0,1), \\ \bigcap_{n\in\mathbb{N}} [0, n/(n+1)) &= [0,1/2).\end{aligned}$$

Definition A.1.10. The Cartesian product of two sets A and B is

$$A \times B = \{(a,b) : a \in A, b \in B\}.$$

In a similar way we define the Cartesian product of $n \in \mathbb{N}$ sets. The repeated Cartesian product of the same set, denoted as

$$A^d = A \times \cdots \times A, \qquad d \in \mathbb{N},$$

is the set of d-tuples or d-dimensional vectors whose components are elements in A.

Example A.1.7. *$\mathbb{R}^d$ is the set of all d dimensional vectors whose components are real numbers.*

Definition A.1.11. A relation R on a set A is a subset of $A \times A$, or in other words a set of pairs of elements in A. If $(a,b) \in R$ we denote $a \sim b$ and if $(a,b) \notin R$ we denote $a \not\sim b$. A relation is reflexive if $a \sim a$ for all $a \in A$. It is symmetric if $a \sim b$ implies $b \sim a$ for all $a, b \in A$. It is transitive if $a \sim b$ and $b \sim c$ implies $a \sim c$ for all $a, b, c \in A$. An equivalence relation is a relation that is reflexive, symmetric, and transitive.

Example A.1.8. *Consider relation 1 where $a \sim b$ if $a \leq b$ over $\mathbb{R}$ and relation 2 where $a \sim b$ if $a = b$ over $\mathbb{Z}$. Relation 1 is reflexive and transitive but not symmetric. Relation 2 is reflexive, symmetric, and transitive and is therefore an equivalence relation.*

[1] One way to rigorously define the set of real numbers is as the completion of the rational numbers. The details may be found in standard real analysis textbooks, for example [37]. We do not pursue this formal definition here.

Definition A.1.12. The sets $U_\alpha, \alpha \in A$ form a partition of U if

$$\biguplus_{\alpha \in A} U_\alpha = U.$$

(see Definition A.1.6.) In other words, the union of the pairwise disjoint sets $U_\alpha, \alpha \in A$ is U. The sets U_α are called equivalence classes.

An equivalence relation $\sim$ on A induces a partition of A as follows: $a \sim b$ if and only if a and b are in the same equivalence class.

Example A.1.9. *Consider the set A of all cities and the relation $a \sim b$ if the cities a, b are in the same country. This relation is reflexive, symmetric, and transitive, and therefore is an equivalence relation. This equivalence relation induces a partition of all cities into equivalence classes consisting of all cities in the same country. The number of equivalence classes is the number of countries.*

A.2 Functions

Definition A.2.1. Let A, B be two sets. A function $f : A \to B$ assigns one element $b \in B$ for every element $a \in A$, denoted by $b = f(a)$.

Definition A.2.2. For a function $f : A \to B$, we define

$$\operatorname{range} f = \{f(a) : a \in A\} \tag{A.1}$$

$$f^{-1}(b) = \{a \in A : f(a) = b\} \tag{A.2}$$

$$f^{-1}(H) = \{a \in A : f(a) \in H\}. \tag{A.3}$$

If $\operatorname{range} f = B$, we say that f is onto. If for all $b \in B$, $|f^{-1}(b)| \leq 1$ we say that f is 1-1 or one-to-one. A function that is both onto and 1-1 is called a bijection.

If $f : A \to B$ is one-to-one, f^{-1} is also a function $f^{-1} : B' \to A$ where $B' = \operatorname{range} f \subset B$. If $f : A \to B$ is a bijection then $f^{-1} : B \to A$ is also a bijection.

Definition A.2.3. Let $f : A \to B$ and $f : B \to C$ be two functions. Their function composition is a function $f \circ g : A \to C$ defined as $(f \circ g)(x) = f(g(x))$.

Proposition A.2.1. *For any function f and any indexed collection of sets $U_\alpha, \alpha \in A$,*

$$f^{-1}\left(\bigcap_{\alpha \in A} U_\alpha\right) = \bigcap_{\alpha \in A} f^{-1}(U_\alpha)$$

$$f^{-1}\left(\bigcup_{\alpha \in A} U_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(U_\alpha)$$

$$f\left(\bigcup_{\alpha \in A} U_\alpha\right) = \bigcup_{\alpha \in A} f(U_\alpha).$$

Proof. We prove the result above for a union and intersection of two sets. The proof of the general case of a collection of sets is similar.

If $x \in f^{-1}(U \cap V)$ then $f(x)$ is in both U and V and therefore $x \in f^{-1}(U) \cap f^{-1}(V)$. If $x \in f^{-1}(U) \cap f^{-1}(V)$ then $f(x) \in U$ and $f(x) \in V$ and therefore $f(x) \in U \cap V$, implying that $x \in f^{-1}(U \cap V)$.

If $x \in f^{-1}(U \cup V)$ then $f(x) \in U \cup V$, which implies $f(x) \in U$ or $f(x) \in V$ and therefore $x \in f^{-1}(U) \cup f^{-1}(V)$. On the other hand, if $x \in f^{-1}(U) \cup f^{-1}(V)$ then $f(x) \in U$ or $f(x) \in V$ implying $f(x) \in U \cup V$ and $x \in f^{-1}(U \cup V)$.

If $y \in f(U \cup V)$ then $y = f(x)$ for some x in either U or V and $y = f(x) \in f(U) \cup f(V)$. On the other hand, if $y \in f(U) \cup f(V)$ then $y = f(x)$ for some x that belongs to either U or V, implying $y = f(x) \in f(U \cup V)$. ∎

Interestingly, the statement $f(U \cap V) = f(U) \cap f(V)$ is *not* true in general.

Example A.2.1. *For $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, we have $f^{-1}(x) = \{\sqrt{x}, -\sqrt{x}\}$ implying that f is not one-to-one. We also have $f^{-1}([1,2]) = [-\sqrt{2}, 1] \cup [1, \sqrt{2}]$.*

Definition A.2.4. Given two functions $f, g : A \to \mathbb{R}$ we denote $f \leq g$ if $f(x) \leq g(x)$ for all $x \in A$, $f < g$ if $f(x) < g(x)$ for all $x \in A$, and $f \equiv g$ if $f(x) = g(x)$ for all $x \in A$.

Definition A.2.5. We denote a sequence of functions $f_n : A \to \mathbb{R}$, $n \in \mathbb{N}$ satisfying $f_1 \leq f_2 \leq f_3 \leq \cdots$ as $f_n \nearrow$.

Definition A.2.6. Given a set $A \subset \Omega$ we define the indicator function $I_A : \Omega \to \mathbb{R}$ as

$$I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \text{otherwise} \end{cases}, \qquad \omega \in \Omega.$$

Example A.2.2. *For any set A, we have $I_A = 1 - I_{A^c}$.*

Example A.2.3. *If $A \subset B$, we have $I_A \leq I_B$.*

A.3 Cardinality

The most obvious generalization of the size of a set A to infinite sets (see Definition A.1.3) implies the obvious statement that infinite sets have infinite size. A more useful generalization can be made by noticing that two finite sets A, B have the same size if and only if there exists a bijection between them, and generalizing this notion to infinite sets.

Definition A.3.1. Two sets A and B (finite or infinite) are said to have the same cardinality, denoted by $A \sim B$ if there exists a bijection between them.

Note that the cardinality concept above defines a relation that is reflexive ($A \sim A$) symmetric ($A \sim B \Rightarrow B \sim A$), and transitive ($A \sim B$ and $B \sim C$ implies $A \sim C$) and is thus an equivalence relation. The cardinality relation thus partitions the set of all sets to equivalence classes containing sets with the same cardinality. For each natural number $k \in \mathbb{N}$, we have an equivalence class containing all finite sets of that size. But there are also other equivalence classes containing infinite sets, the most important one being the equivalence class that contains the natural numbers $\mathbb{N}$.

Definition A.3.2. Let A be an infinite set. If $A \sim \mathbb{N}$ then A is a countably infinite set. If $A \not\sim \mathbb{N}$ then A is a uncountably infinite set.

Proposition A.3.1. *Every infinite subset E of a countably infinite set A is countably infinite.*

Proof. A is countably infinite, so we can construct an infinite sequence $x_1, x_2, \ldots$ containing the elements of A. We can construct another sequence, $y_1, y_2, \ldots$, that is obtained by omitting from the first sequence the elements in $A \setminus E$. The sequence $y_1, y_2, \ldots$ corresponds to a bijection between the natural numbers and E. ∎

Proposition A.3.2. *A countable union of countably infinite sets is countably infinite.*

Proof. Let $A_n, n \in \mathbb{N}$ be a collection of countably infinite sets. We can arrange the elements of each A_n as a sequence that forms the n-row of a table with infinite rows and columns. We refer to the element at the i-row and j-column in that table as A_{ij}. Traversing the table in the following order: A_{11}, A_{21}, A_{12}, A_{31}, A_{22}, A_{13}, and so on (traversing south-west to north-east diagonals of the table starting at the north-west corner), we express the elements of $A = \cup_{n \in \mathbb{N}} A_n$ as an infinite sequence. That sequence forms a bijection from $\mathbb{N}$ to A. ∎

Corollary A.3.1. *If A is countably infinite then so is A^d, for $d \in \mathbb{N}$.*

Proof. If A is countably infinite, then $A \times A$ corresponds to one copy of A for each element of A, and thus we have a bijection between A^2 and a countably infinite union of countably infinite sets. The previous proposition implies that A^2 is countably infinite. The general case follows by induction. ∎

It can be shown that countably infinite sets are the "smallest" sets (in terms of the above definition of cardinality) among all infinite sets. In other words, if A is uncountably infinite, then there exists an onto function $f : A \to \mathbb{N}$ but no onto function $f : \mathbb{N} \to A$.

Proposition A.3.3. *Assuming $a < b$ and $d \in \mathbb{N}$,*

$$\begin{aligned}
&\mathbb{N} \sim \mathbb{Z} \sim \mathbb{Q}\\
&\mathbb{N} \not\sim [a, b]\\
&\mathbb{N} \not\sim (a, b)\\
&\mathbb{N} \not\sim \mathbb{R}\\
&\mathbb{N} \not\sim \mathbb{R}^d.
\end{aligned}$$

We first define the concept of a binary expansion of a number, which will be used in the proof of the proposition below.

Definition A.3.3. The binary expansion of a number $r \in [0,1]$ is defined as $0.b_1 b_2, b_3 \dots$ where $b_n \in \{0,1\}, n \in \mathbb{N}$ and

$$r = \sum_{n\in\mathbb{N}} b_n 2^{-n}.$$

Example A.3.1. *The binary expansions of* $1/4$ *is* 0.01 *and the binary expansion of* $3/4$ *is* $0.11 = 1/2 + 1/4$.

Proof. Obviously there exists a bijection between the natural numbers and the natural numbers. The mapping $f : \mathbb{N} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} (n-1)/2 & n \text{ is odd} \\ -n/2 & n \text{ is even} \end{cases}$$

maps 1 to 0, 2 to -1, 3 to 1, 4 to -2, 5 to 2, etc., and forms a bijection between $\mathbb{N}$ and $\mathbb{Z}$.

A rational number is a ratio of a numerator and a denominator integers (note however that two pairs of numerators and denominators may yield the same rational number). We thus have that $\mathbb{Q}$ is a subset of $\mathbb{Z}^2$, which is countably infinite by Proposition A.3.2. Using Proposition A.3.1, we have that $\mathbb{Q}$ is countably infinite.

We next show that there does not exist a bijection between the natural numbers and the interval $[0,1]$. If there was such a mapping f, we could arrange the numbers in $[0,1]$ as a sequence $f(n), n \in \mathbb{N}$ and form a table A with infinite rows and columns where column n is the binary expansion of the real number $f(n) \in [0,1]$.

We could then create a new real number whose binary expansion is $b_n, n \in \mathbb{N}$ with $b_n \neq A_{nn}$ for all $n \in \mathbb{N}$ by traversing the diagonal of the table and choosing the alternative digits. This new real number is in $[0,1]$ since it has a binary expansion, and yet it is different from any of the columns of A. We have thus found a real number that is different from any other number[2] in the range of f, contradicting the fact that f is onto.

Since there is no onto function from the naturals to $[0,1]$ there can be no onto function from the natural numbers to $\mathbb{R}$ or its Cartesian products $\mathbb{R}^d$. ∎

We extend below the definition of a Cartesian product (Definition A.1.10) to an infinite number of sets.

Definition A.3.4. Let A, T be sets. The notation A^T denotes a Cartesian product of multiple copies of A, one copy for each element of the set T. In other words, A^T is the set of all functions from T to A. The notation A^∞ denotes $A^{\mathbb{N}}$, a product of a countably infinite copies of A.

[2] The setup described above is slightly simplified. A rigorous proof needs to resolve the fact that some numbers in $[0,1]$ have two different binary expansions, for example, $0.0111111\ldots = 0.1$. See for example [37].

Example A.3.2. *The set $\mathbb{R}^\infty$ is the set of all infinite sequences over the real line $\mathbb{R}$*

$$\mathbb{R}^\infty = \{(a_1, a_2, a_2, \ldots,) : a_n \in \mathbb{R} \text{ for all } n \in \mathbb{N}\}$$

and the set $\{0,1\}^\infty$ is the set of all infinite binary sequences. The set $R^{[0,1]}$ is a Cartesian product of multiple copies of the real numbers — one copy for each element of the interval $[0,1]$, or in other words the set of all functions from $[0,1]$ to $\mathbb{R}$.

Example A.3.3. *The set $\{0,1\}^A$ is the set of all functions from A to $\{0,1\}$, each such function implying a selection of an arbitrary subset of A (the selected elements are mapped to 1 and the remaining elements are mapped to 0). A similar interpretation may given to sets of size 2 that are different than $\{0,1\}$. Recalling Definition A.1.7, we thus have if $|B| = 2$, then B^A corresponds to the power set 2^A, justifying its notation.*

A.4 Limits of Sets

Definition A.4.1. For a sequence of sets A_n, $n \in \mathbb{N}$, we define

$$\inf_{k \geq n} A_k = \bigcap_{k=n}^{\infty} A_k$$
$$\sup_{k \geq n} A_k = \bigcup_{k=n}^{\infty} A_k$$
$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \inf_{k\geq n} A_k = \bigcup_{n\in\mathbb{N}} \bigcap_{k=n}^{\infty} A_k$$
$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \sup_{k\geq n} A_k = \bigcap_{n\in\mathbb{N}} \bigcup_{k=n}^{\infty} A_k.$$

Applying De-Morgan's law (Proposition A.1.1) we have

$$\left(\liminf_{n\to\infty} A_n\right)^c = \limsup_{n\to\infty} A_n^c.$$

Definition A.4.2. If for a sequence of sets A_n, $n \in \mathbb{N}$, we have $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$, we define the limit of A_n, $n \in \mathbb{N}$ to be

$$\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n,$$

The notation $A_n \to A$ is equivalent to the notation $\lim_{n\to\infty} A_n = A$.

Example A.4.1. *For the sequence of sets $A_k = [0, k/(k+1))$ from Example A.1.6* *we have*

$$\inf_{k\geq n} A_k = [0, n/(n+1))$$
$$\sup_{k\geq n} A_k = [0,1)$$
$$\limsup_{n\to\infty} A_n = [0,1)$$
$$\liminf_{n\to\infty} A_n = [0,1)$$
$$\lim A_n = [0,1).$$

We have the following interpretation for the lim inf and lim sup limits.

Proposition A.4.1. *Let $A_n, n \in \mathbb{N}$ be a sequence of subsets of Ω. Then*

$$\limsup_{n\to\infty} A_n = \left\{\omega \in \Omega \,:\, \sum_{n\in\mathbb{N}} I_{A_n}(\omega) = \infty\right\}$$
$$\liminf_{n\to\infty} A_n = \left\{\omega \in \Omega \,:\, \sum_{n\in\mathbb{N}} I_{A_n^c}(\omega) < \infty\right\}.$$

In other words, $\limsup_{n\to\infty} A_n$ *is the set of* $\omega \in \Omega$ *that appear infinitely often (abbreviated i.o.) in the sequence* A_n*, and* $\liminf_{n\to\infty} A_n$ *is the set of* $\omega \in \Omega$ *that always appear in the sequence* A_n *except for a finite number of times.*

Proof. We prove the first part. The proof of the second part is similar. If $\omega \in \limsup_{n\to\infty} A_n$ then by definition for all n there exists a k_n such that $\omega \in A_{k_n}$. For that ω we have $\sum_{n\in\mathbb{N}} I_{A_n}(\omega) = \infty$. Conversely, if $\sum_{n\in\mathbb{N}} I_{A_n}(\omega) = \infty$, there exists a sequence $k_1, k_2, \ldots$ such that $\omega \in A_{k_n}$, implying that for all $n \in \mathbb{N}$, $\omega \in \cup_{i\geq n} A_i$. ∎

Corollary A.4.1.

$$\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n.$$

Definition A.4.3. A sequence of sets $A_n, n \in \mathbb{N}$ is monotonic non-decreasing if $A_1 \subset A_2 \subset A_3 \subset \cdots$ and monotonic non-increasing if $\cdots \subset A_3 \subset A_2 \subset A_1$. We denote this as $A_n \nearrow$ and $A_n \searrow$, respectively. If $\lim A_n = A$, we denote this as $A_n \nearrow A$ and $A_n \searrow A$, respectively.

Proposition A.4.2. *If* $A_n \nearrow$ *then* $\lim_{n\to\infty} A_n = \cup_{n\in\mathbb{N}} A_n$ *and if* $A_n \searrow$ *then* $\lim_{n\to\infty} A_n = \cap_{n\in\mathbb{N}} A_n$.

Proof. We prove the first statement. The proof of the second statement is similar. We need to show that if A_n is monotonic non-decreasing, then $\limsup A_n = \liminf A_n = \cup_n A_n$. Since $A_i \subset A_{i+1}$, we have $\cap_{k\geq n} A_k = A_n$, and

$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k = \bigcup_{n\in\mathbb{N}} A_n$$
$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k \subset \bigcup_{k\in\mathbb{N}} A_k = \liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n.$$

∎

The following corollary of the above proposition motivates the notations lim inf and lim sup.

Corollary A.4.2. *Since* $B_n = \cup_{k \geq n} A_k$ *and* $C_n = \cap_{k \geq n} A_k$ *are monotonic sequences*

$$\liminf_{n \to \infty} A_n = \lim_{n \to \infty} \inf_{k \geq n} A_n$$
$$\limsup_{n \to \infty} A_n = \lim_{n \to \infty} \sup_{k \geq n} A_n.$$

A.5 Notes

Most of the material in this section is standard material in set theory. More information may be found in any set theory textbook. A classic textbook is [20]. Limits of sets are usually described at the beginning of measure theory or probability theory textbooks, for example [5, 1, 33].

A.6 Exercises

1. Prove the assertions in Example A.1.1.
2. Prove the assertion in Example A.1.6.
3. Prove that $f(U \cap V) = f(U) \cap f(V)$ is not true in general.
4. Let $A_0 = \{a\}$ and define $A_k = 2^{A_{k-1}}$ for $k \in \mathbb{N}$. Write down the elements of the sets A_k for all $k = 1, 2, 3$.
5. Let A, B, C be three finite sets. Describe intuitively the sets $A^{(B^C)}$ and $(A^B)^C$. What are the sizes of these two sets?
6. Let A be a finite set and B be a countably infinite set. Are the sets A^∞ and B^∞ countably infinite or uncountably infinite?
7. Find a sequence of sets A_n, $n \in \mathbb{N}$ for which $\liminf A_n \neq \limsup A_n$.
8. Describe an equivalence relation with an uncountably infinite set of equivalence classes, each of which is a set of size 2.